م 1+2

Vectors & Scalars

Vectors

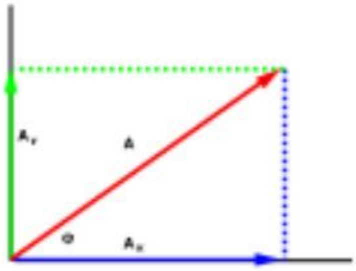
❑ A scalar can be completely defined by *magnitude*.	❑ A vector needs both *magnitude* and *direction* to be defined.
❑ Examples: mass, density, volume, and energy.	❑ Examples: displacement, velocity, acceleration, and force.
❑ Its value is independent of any chosen coordinates.	❑ Its value is coordinate system dependent.
❑ Mathematically, scalars obey the normal algebraic rules of addition, multiplication ...etc.	❑ Mathematically, vectors need a special treatment known as ***Vectors Algebra***.

Basic method to describe vector is Cartesian coordinate system.

Vector Algebra

A given vector **A** can be specified by:

Its ***magnitude*** (A) and its ***direction*** (ϕ) relative to some chosen coordinate system.



The set of its ***components***, or projections onto the coordinate axes ; Since

$$A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ ***The magnitude***

and

$$\phi = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$ ***The direction***

Vector Addition:

The addition of two vectors is defined by the equation:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z)$$

Note that:

A+B = B+A **Commutative Law**

A+(B+C) = (A+B)+C **Associative Law**

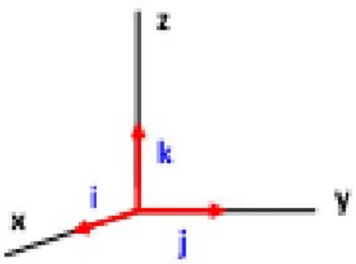
The Unit Vectors:

A unit vector is a vector whose magnitude is **unity**. Unit vectors are often assigned by the symbol **e**, The three unit vectors

$\mathbf{e}_x = (1,0,0)$ $\mathbf{e}_y = (0,1,0)$ $\mathbf{e}_z = (0,0,1)$

for **Cartesian coordinates**

$\mathbf{e}_x = \mathbf{i}$ $\mathbf{e}_y = \mathbf{j}$ $\mathbf{e}_z = \mathbf{k}$



The Scalar Product:

Given two vectors **A** and **B**, the ***scalar or " dot " product***, **A.B** is the scalar defined by the equation

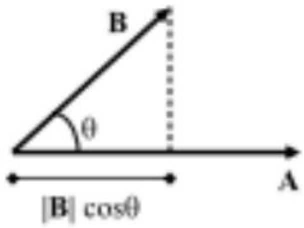
$$\mathbf{A.B} = A_xB_x + A_yB_y + A_zB_z$$

Main properties of the scalar product:

- It is **commutative**, **A.B = B.A**
- It is **distributive**, **A . (B + C) = A . B + A . C**
- It is a **scalar**. If vector **A** is expressed as $(A_x, 0, 0)$ and the vector **B** as $(B_x, B_y, 0)$ or $(B\cos\theta, B\sin\theta, 0)$, then,

$$\mathbf{A.B} = A_xB_x = A(B\cos\theta) = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

- Hence, The geometrical interpretation of **A . B** is that it is the projection of **B** onto **A** times the length of **A**.
- If **A. B** $= 0$, and neither **A** nor **B** is null, then ($\cos\theta = 0$) and **A** is ⊥ to **B**.



Similarly;

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$
$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$

The Vector Product

Given two vectors **A** and **B**, the ***vector or "cross " product***, **A** x **B** is a **vector** whose components are given by the equation

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y\,,\ \ A_zB_x - A_xB_z\,,\ \ A_xB_y - A_yB_x\,)$$

which is equal to the *determinant*,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \vec{j}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \vec{i}(A_yB_z - A_zB_y) + \vec{j}(A_zB_x - A_xB_z) + \vec{k}(A_xB_y - A_yB_x)$$

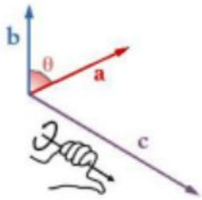
Main properties of the *vector product*:

- It is ***anti-commutative***, $\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$
- It is ***distributive***, $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
- the resultant is a ***vector*** . Its ***magnitude*** is given by;

$$|\,\mathbf{A} \times \mathbf{B}\,| = |A|\ |B|\ \sin\theta$$

where θ is the **smallest** angle between **A** and **B**.

The Vector Product



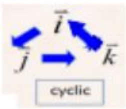
❑ The ***direction*** of the resultant vector is ⊥ to the plane containing **A** and **B**.

Hence,

$$\mathbf{A} \times \mathbf{B} = (A\,B \sin\theta)\,\mathbf{n}$$

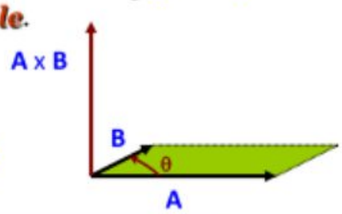
where **n** is a unit vector normal to the plane containing **A** and **B**. The sense of **n** is given by the ***right –hand rule***.

Therefore,



$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

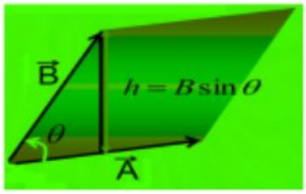
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$



The *cross product* $\mathbf{A} \times \mathbf{B}$ has

1- A magnitude of $A\,B \sin\theta$ which is equal to the area of the parallelogram with sides **A** and **B** shown by the shaded area in the Figure.

2- A direction ⊥ to the plane containing **A** and **B**.



Derivative of a vector

Previously we were concerned mainly with vector algebra. Now, we will begin to study the calculus of vectors and its use in the description of the motion of particles.

Consider a vector **A**, whose components are functions of a single variable u which is usually the time t, i.e,

$$\mathbf{A}(u) = \mathbf{i}\,A_x(u) + \mathbf{j}\,A_y(u) + \mathbf{k}\,A_z(u)$$

The derivative of **A** with respect to u is defined by

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}$$

The rules for differentiating vector products

$$\frac{d}{du}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \qquad \frac{d}{du}(\mathbf{A}.\mathbf{B}) = \frac{d\mathbf{A}}{du}.\mathbf{B} + \mathbf{A}.\frac{d\mathbf{B}}{du}$$

$$\frac{d}{du}(n\mathbf{A}) = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du} \qquad \frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{du}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{du}$$

Notice that it is necessary to maintain the order of the terms in the derivative of the cross product.

The position vector

Velocity & Acceleration In rectangular coordinates

In a given reference system, ***the position of a particle*** can be specified by a single vector. This vector is called *the position vector* of the particle.

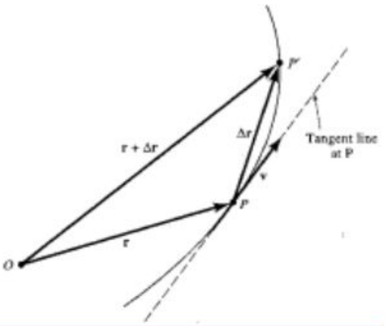
In rectangular coordinates (cartesian coordinates), the *position vector* is simply

$$\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$$

The velocity vector

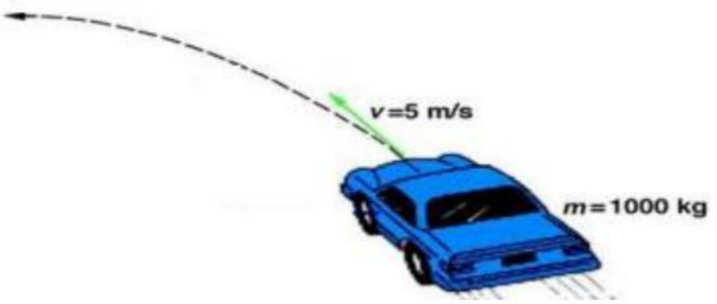
For a moving particle, these components are functions of the time.
The time derivative of r *is* called the ***velocity*, (v)**, which is given by:
$(\dot{x}=dx/dt$, $\dot{y}=dy/dt$ and $\dot{z}=dz/dt)$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\,\dot{x} + \mathbf{j}\,\dot{y} + \mathbf{k}\,\dot{z}$$



The vector $d\mathbf{r}/dt$ expresses both the *direction* and the *rate* of motion. As Δt approaches zero, the point P' approaches P, and the direction of the vector $\Delta\mathbf{r}/\Delta t$ approaches the direction of the tangent to the path at P, which is $d\mathbf{r}/dt$.

Note: The velocity vector is always tangent to the path of motion.



The **magnitude** of the velocity is called the *speed* (v). In rectangular components the speed is just

$$v = \frac{ds}{dt} = |\mathbf{v}| = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)^{1/2}$$

where s is the distance.

The acceleration vector

The time derivative of the velocity is called the ***acceleration*** (**a**). Hence;

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{i}\,\ddot{x} + \mathbf{j}\,\ddot{y} + \mathbf{k}\,\ddot{z}$$

EXAMPLE 1.10.1

Projectile Motion

Let us examine the motion represented by the equation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity $\mathbf{v}$ is obtained by differentiating with respect to t, namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The acceleration, likewise, is given by

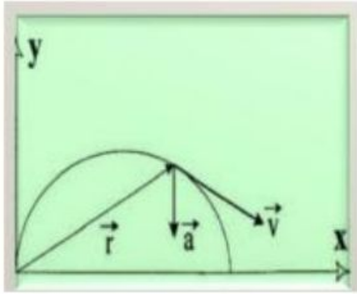
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, $\mathbf{a}$ is in the negative y direction and has the constant magnitude g. The path of motion is a parabola, as shown in Figure 1.10.3. The speed v varies with t according to the equation

$$v = [b^2 + (c - gt)^2]^{1/2}$$

Exp(1-10-1):**Projectile Motion**

Examples of Velocity & Acceleration In rectangular coordinates



The position vector	$\vec{r} = i\, bt + j\, (ct - \frac{gt^2}{2})$
The velocity	$\vec{v} = i\, b + j\, (c - gt)$
The acceleration	$\vec{a} = -j\, g$
The path	**Parabola**

EXAMPLE 1.10.2

Circular Motion

Suppose the position vector of a particle is given by

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating **r**, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin \omega t - \mathbf{j}b\omega^2 \cos \omega t$$

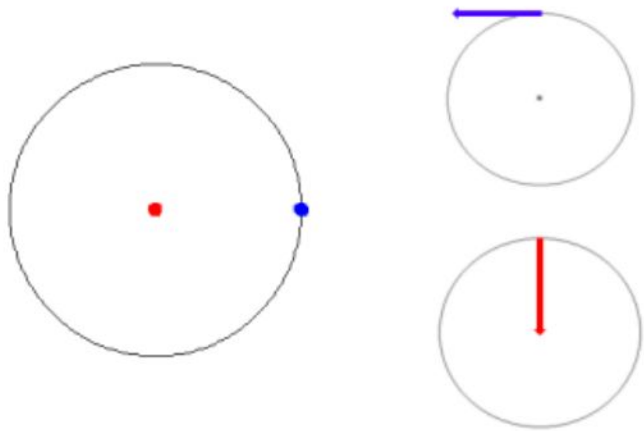
In this case the acceleration is perpendicular to the velocity, because the dot product of **v** and **a** vanishes:

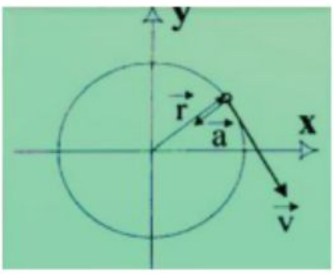
$$\mathbf{v} \cdot \mathbf{a} = (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) + (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) = 0$$

Comparing the two expressions for **a** and **r**, we see that we can write

$$\mathbf{a} = -\omega^2 \mathbf{r}$$

Exp(1-10-2):
Circular Motion





The position vector	$\mathbf{r} = \mathbf{i}\, b\cos\omega t + \mathbf{j}\, b\sin\omega t$
The velocity	$\mathbf{v} = -\mathbf{i}\, b\omega\sin\omega t + \mathbf{j}\, b\omega\cos\omega t$
The acceleration	$\mathbf{a} = -\mathbf{i}\, b\omega^2\cos\omega t - \mathbf{j}\, b\omega^2\sin\omega t$ $\mathbf{a} = -\omega^2\,\mathbf{r}$ $\quad v \perp a$
The path	**Circle**

H.W. Please make sure that **v.a**=0

الفصل الثاني

Chapter 2

حركة جسم في بعدين او ثلاثة ابعاد

Motion a partical in Two or Three Dimensions

نفرض ان المتجه $\vec{A}(t)$ دالة للزمن اي ان

$$\vec{A}(t) = \hat{x}\vec{A}_x(t) + \hat{y}\vec{A}_y(t) + \hat{z}\vec{A}_z(t)$$

حيث ان $\hat{x}$ و $\hat{y}$ و $\hat{z}$ متجهات احادية

$$so\ \frac{d\vec{A}}{dt} = \hat{x}\frac{dA_x}{dt} + \hat{y}\frac{dA_y}{dt} + \hat{z}\frac{dA_z}{dt}$$

ان تفاضل المتجهات له الخواص التالية

1.) $\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$

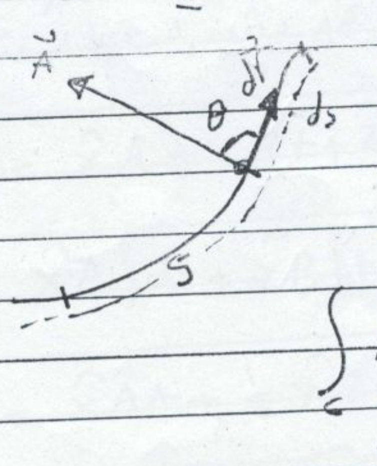
2.) $\frac{d}{dt}(F(t)A(t)) = \frac{dF}{dt}\vec{A} + F\frac{d\vec{A}}{dt}$

3.) $\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$ [ضرب عددي]

4.) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$ [ضرب اتجاهي]

F is function.

نفرض $\vec{A}$ دالة متجه ومعرفة في نقطة ما على المنحني C وكالاتي



اذن التكامل الخطي لهذه الدالة المتجهة A يمثل بالمعادلة التالية

$$\int_c \vec{A}\cdot d\vec{r} = \int_c A\cos\theta\, ds \quad (1)$$

حيث ان ds يمثل مسافة صغيرة على المنحني C و dr هو متجه الموضع لهذه القطعة الصغيرة ds .

مثال على ذلك الشغل المنجز على القوة $\vec{F}$ والذي يتغير من نقطة الى اخرى حسب حركة الجسم على المنحني C يعطى بالمعادلة التالية

$$W = \int_C \vec{F} \cdot d\vec{r} \quad \text{—— (2)}$$

من الممكن كتابة التكامل بالمعادلة (2) على النحو التالي

$$\vec{r} = \hat{x}\,x + \hat{y}\,y + \hat{z}\,z$$

$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\therefore \int_C \vec{A} \cdot d\vec{r} = \int (A_x dx + A_y dy + A_z dz)$$

or

$$\int_C \vec{A} \cdot d\vec{r} = \int \left(\vec{A} \cdot \frac{d\vec{r}}{ds}\right) ds = \int \left(A_x \frac{dx}{ds} + A_y \frac{dy}{ds} + A_z \frac{dz}{ds}\right) ds$$

مثال : اذا علمت ان متجه السرعة $\vec{v}$ لجسم متحرك هو

$$\vec{v} = \hat{x}A + \hat{y}Bt + \hat{z}Ct^{-1}$$

حيث A و B و C ثوابت، اوجد متجه الموقع لهذا الجسم

الحل :

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}A + \hat{y}Bt + \hat{z}Ct^{-1}$$

$$\int d\vec{r} = \hat{x}A\int dt + \hat{y}B\int t\,dt + \int \hat{z}C\,t^{-1}\,dt$$

$$\vec{r} = \hat{x}At + \frac{1}{2}\hat{y}Bt^2 + \hat{z}C\ln t + r_0$$

حيث r_0 ثابت التكامل .

مثال٢/ جسم متحرك متجه الموضع له يمثل بالعلاقة التالية

$$\vec{r}(t) = \hat{x}bt + \hat{y}(ct - \tfrac{1}{2}gt^2)$$

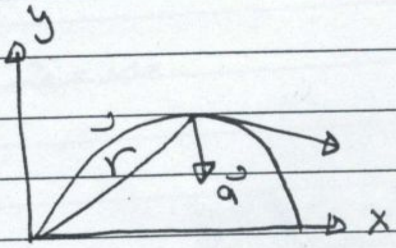
حيث b و c ثوابت، و g تعجيل ارضي . جد ما يلي (سرعة) الجسم $\vec{v}$. انطلاقات الجسم $|\vec{v}|$ ، تعجيل الجسم $\vec{a}$. ناقش حركة الجسم؟

الحل/

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}b + \hat{y}(c - gt)$$

$$|\vec{v}| = \sqrt{b^2 + (c-gt)^2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -g\hat{y}$$



بما ان مركبة السرعة باتجاه z تساوي صفر اذن الحركة في المستوي xy وبما ان التعجيل باتجاه معاكس للمحور $\hat{y}$ ومقداره g وبما ان التعجيل ثابت وبما ان السرعة لها علاقة مع الزمن فان حركة الجسم تمثل حركة قذائفية.

مثال٣/ جسم متحرك توصف بالمعادلة التالية

$$\vec{r} = \hat{x}b\sin\omega t + \hat{y}b\cos\omega t + \hat{z}c$$

حيث ان b و c ثوابت، ω السرعة الزاوية، جد ما يلي

١- بعد الجسم عن نقطة الاصل $|\vec{r}|$

٢- سرعة الجسم $\vec{v}$

٣- انطلاقات الجسم $|\vec{v}|$

٤- تعجيل الجسم $\vec{a}$

٥- برهن ان السرعة $\vec{v}$ عمودية على التعجيل $\vec{a}$ $(\vec{v}\perp\vec{a})$

٦- = = التعجيل $\vec{a}$ = على المحور z $(\vec{a}\perp\hat{z})$

٧- برهن ان مسار الجسم دائري نصف قطره b وموازي للمستوي xy ويبتعد عن نقطة الاصل مسافة c .

الحل/

$$|\vec{r}| = \sqrt{b^2\sin^2 wt + b^2\cos^2 wt + c^2}$$

$$|\vec{r}| = \sqrt{b^2 + c^2}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}wb\cos wt - \hat{y}bw\sin wt$$

$$|\vec{v}| = \sqrt{b^2w^2\cos^2 wt + b^2w^2\sin^2 wt} = \sqrt{b^2w^2} = bw$$

ملاحظة

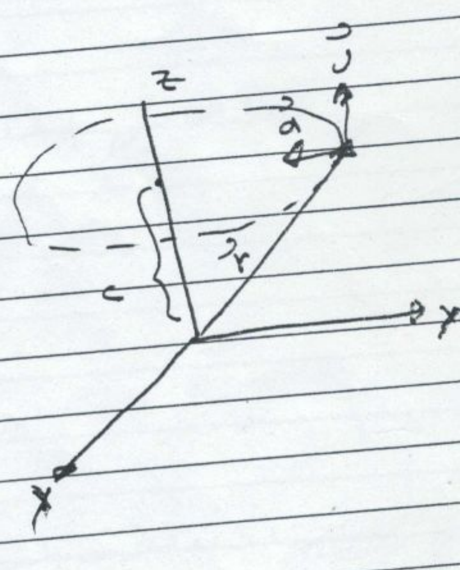
$$\vec{a}\cdot\vec{v} = 0 \Rightarrow \vec{a} \perp \vec{v}$$

$$\vec{a} = -\hat{x}w^2 b\sin wt - \hat{y}bw^2\cos wt$$

$$\vec{a}\cdot\vec{v} = -b^2w^3\sin wt\cos wt + b^2w^3\sin wt\cos wt$$

$$= 0$$

$$\therefore \vec{a} \perp \vec{v}$$

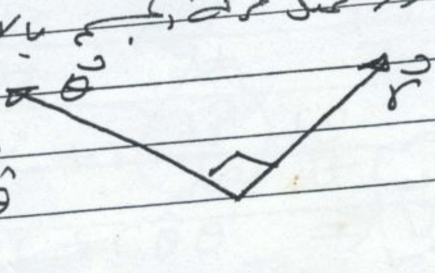


$$\vec{a}\cdot\hat{z} = 0 \quad \therefore \vec{a} \perp \hat{z}$$

بما ان السرعة في المستوي (x,y) وبالانطلاق (bw)
اذن الحركة دائرية في المستوي (xy) ونصف قطرها (b)
ومن الشكل نلاحظ ان المسار يرتفع عن نقطة الاصل
بمقدار (c).

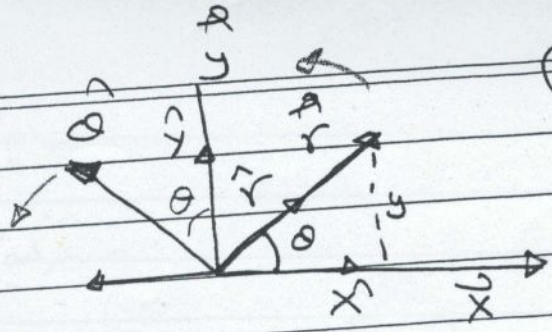
* حركة الجسم في مستوي (الاحداثيات القطبية)

س/ عرف نظام الاحداثيات القطبية المستوية (r,θ) واشتق علاقة لسرعة الجسم وتعجيله لهذا النظام مبيناً مركبات كل من السرعة والتعجيل في الشكل اعلاه، تمثل حركة الجسم بالاحداثيات القطبية (r,θ) وكما يلي.



$\hat{r}$ = متجه احادي باتجاه زيادة r وعمودي على $\hat{\theta}$

$\hat{\theta}$ 〃 〃 θ 〃 〃 $\hat{r}$



عين نتائج المحاور (r,θ) بدلالة الاحداثيات الديكارتية (x,y) متعامدة

$$x = r\cos\theta$$
$$y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \sin^{-1}\left(\frac{y}{r}\right) = \cos^{-1}\left(\frac{x}{r}\right)$$

هناك علاقة بين $\hat{x}$ و $\hat{y}$ و $\hat{r}$ و $\hat{\theta}$ وكالاتي

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$
$$\hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta$$

نتفاضل هاتين المعادلتين

$$\frac{d\hat{r}}{d\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\hat{x}\cos\theta - \hat{y}\sin\theta = -\hat{r}$$

$$\therefore \boxed{\frac{d\hat{r}}{d\theta} = \hat{\theta}}, \quad \boxed{\frac{d\hat{\theta}}{d\theta} = -\hat{r}}$$

اذن متجه الموضع r يكتب بالاحداثيات القطبية وكالاتي

$$\boxed{\vec{r} = \hat{r}(\theta)\, r}$$

$$\bar{r} = r\cdot|\bar{r}| = r\cdot e_r$$
$$e_r\frac{dr}{dt} + r\cdot\frac{\partial e_r}{\partial t}$$

من هذه العلاقة نحصل على سرعة جسم

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt} \qquad \frac{d\bar{e_r}}{dt} = \frac{d\bar{e_r}}{\partial\theta}\cdot\frac{\partial\theta}{\partial t}$$

$$= \hat{r}\dot{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$

$$= \hat{r}\dot{r} + r\hat{\theta}\dot{\theta} \Rightarrow \boxed{\vec{v} = \hat{r}v_r + \hat{\theta}v_\theta}$$

$$\vec{v} = \hat{r}v_r +$$

$$\begin{cases} v_r = \dot{r} & \text{مركبة قطبية} \\ v_\theta = r\dot{\theta} & \text{مركبة مستعرضة} \end{cases}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{r}\ddot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{\theta} r\ddot{\theta} + \hat{\theta}\dot{\theta}\dot{r} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \hat{r}\ddot{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \hat{\theta} r\ddot{\theta} + \hat{\theta}\dot{\theta}\dot{r} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}$$

$$= \hat{r}\ddot{r} + \dot{r}\hat{\theta}\dot{\theta} + \hat{\theta} r\ddot{\theta} + \hat{\theta}\dot{\theta}\dot{r} - r\dot{\theta}\hat{r}\dot{\theta}$$

$$= \hat{r}(\ddot{r} - r\dot{\theta}^2) + \hat{\theta}[2\dot{\theta}\dot{r} + r\ddot{\theta}]$$

$$\boxed{a = \hat{r}\, a_r + \hat{\theta}\, a_\theta}$$

حيث ان

$$\begin{cases} a_r = (\ddot{r} - r\dot{\theta}^2) & \text{مركبة قطبية} \\ a_\theta = (2\dot{\theta}\dot{r} + r\ddot{\theta}) & \text{مركبة مستعرضة} \end{cases}$$

$r\dot{\theta}^2 = \frac{v_\theta^2}{r}$ يمثل التعجيل الجاذب

$2\dot{r}\dot{\theta}$ ويمثل تعجيل كريوليس

اذا كانت الحركة على دائرة ففي هذه الحالة

$r = const.$

$\dot{r} = \ddot{r} = 0$

$$\left.\begin{aligned} a_r &= -r\dot{\theta}^2 \Longrightarrow -\frac{v_\theta^2}{r} \\ a_\theta &= r\ddot{\theta} \end{aligned}\right\} \text{الحركة الدائرية}$$

الحركة في ثلاث اتجاهات

Motion in Three Dimensions

(1) الإحداثيات الديكارتية (x, y, z)

ان السرعة والتعجيل في هذه الإحداثيات تكتب بالشكل التالي

$$\vec{r} = \hat{x}\,x + \hat{y}\,y + \hat{z}\,z$$

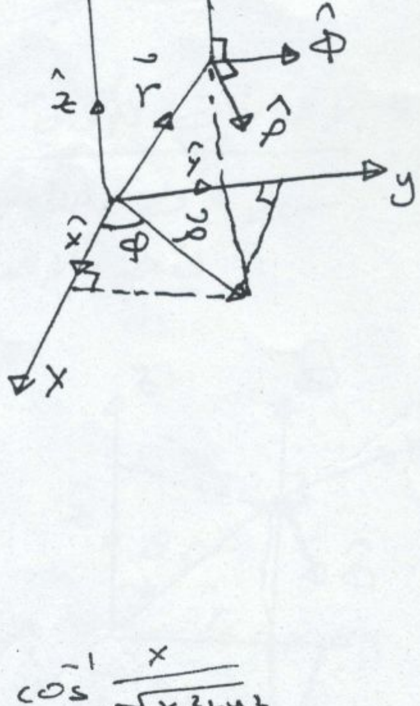
$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt}$$

$$\vec{v} = \hat{x}\,v_x + \hat{y}\,v_y + \hat{z}\,v_z$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{x}\,a_x + \hat{y}\,a_y + \hat{z}\,a_z$$

سنعرف نظام الإحداثيات القطبية الأسطوانية (ρ, ϕ, z) وأنشئ علاقة السرعة للجسم والتعجيل في هذا النظام ؟

من الممكن تمثيل حركة الجسم بالإحداثيات الأسطوانية (ρ, ϕ, z) على النحو التالي.



$\hat{\rho}$ = متجه احادي اتجاهه باتجاه زيادة ρ وعمودي على محور $\hat{z}$ ويقع في المستوي xy

$\hat{\phi}$ = = = = = ϕ = = = عمودي على $\hat{\rho}$ و $\hat{z}$

$\hat{z}$ = = = = = z = = = على $\hat{\rho}$ و $\hat{\phi}$

من الشكل نلاحظ

$$X = \rho\cos\phi$$

$$Y = \rho\sin\phi$$

$$z = z$$

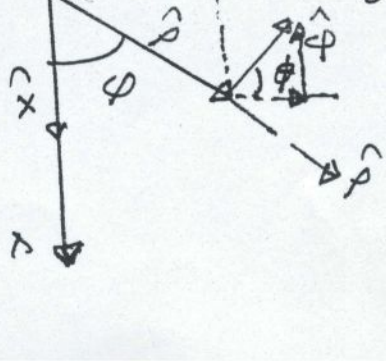
$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}} = \cos^{-1}\frac{x}{\sqrt{x^2+y^2}}$$

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

ملاحظة : $\hat{\rho}, \hat{\phi} \Rightarrow$ دالة للزاوية ϕ



من هاتين المعادلتين نحصل على

$$\frac{d\hat{\rho}}{d\phi} = \hat{\phi}$$

$$\frac{d\hat{\phi}}{d\phi} = -\hat{\rho}$$

لغرض حساب السرعة والتعجيل نلاحظ ان صيغة الموضع في هذه الاحداثيات يمكن كتابتها بالشكل التالي

$$\vec{r} = \hat{\rho}\rho + \hat{z}z$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{\rho}\dot{\rho} + \rho\frac{d\hat{\rho}}{dt} + \hat{z}\dot{z} + z\frac{d\hat{z}}{dt} = 0$$ ثابت

$$\vec{v} = \hat{\rho}\dot{\rho} + \rho\frac{d\hat{\rho}}{d\phi}\frac{d\phi}{dt} + \hat{z}\dot{z}$$

$$\boxed{\vec{v} = \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}}$$

بنفس الطريقة يمكن حساب التعجيل

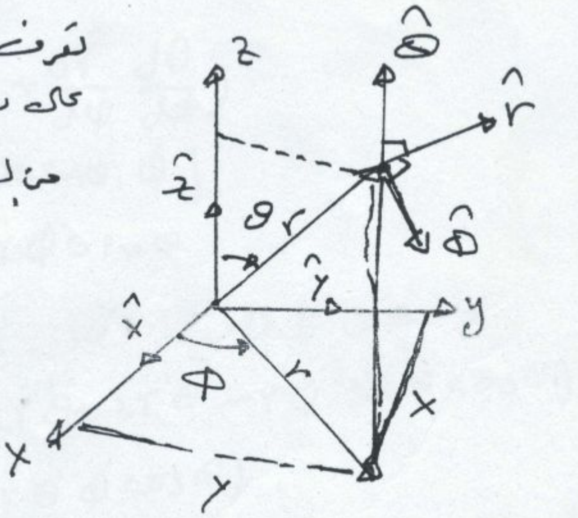
$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\boxed{\vec{a} = \hat{\rho}(\ddot{\rho} - \rho\dot{\phi}^2) + \hat{\phi}(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) + \hat{z}\ddot{z}}$$

* الاحداثيات الكروية : (r, θ, ϕ)

سوف نعرف الاحداثيات الكروية ثم نشتق علاقة السرعة والتعجيل في هذه الاحداثيات .

نعرف المتجهات الاحادية $\hat{r}$ و $\hat{\theta}$ و $\hat{\phi}$ بأنها متجهات متعامدة على بعضها البعض واتجاهها باتجاه تزايد r, θ, ϕ على التوالي من الشكل نلاحظ



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\hat{r} = \hat{z}\cos\theta + \hat{\rho}\sin\theta$$
$$= -\hat{z}\cos\theta + (\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi)$$

$$\hat{\theta} = -\hat{z}\sin\theta + \hat{\rho}\cos\theta = -\hat{z}\sin\theta + (\hat{x}\cos\phi\cos\theta + \hat{y}\sin\phi\cos\theta)$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

تفاضل هذه المتجهات (الاحداثيات الاحادية) فنحصل على

$$\frac{d\hat{r}}{d\theta} = \hat{\theta} \quad , \quad \frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}\sin\theta$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} \quad , \quad \frac{\partial \hat{\theta}}{\partial \phi} = \hat{\phi}\cos\theta$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0 \quad , \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho} = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta)$$

لايجاد السرعة والتعجيل نكتب اولاً متجه الموضع في هذه الاحداثيات وعلى النحو التالي.

$$\vec{r} = \hat{r}(\theta, \phi)\, r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{r}\dot{r} + r\frac{d\hat{r}}{dt}$$
$$= \hat{r}\dot{r} + r\left(\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{r}}{d\phi}\frac{d\phi}{dt}\right)$$
$$= \hat{r}\dot{r} + r[\hat{\theta}\dot{\theta} + \hat{\phi}\sin\theta\,\dot{\phi}]$$
$$\vec{v} = \hat{r}\dot{r} + \hat{\theta} r\dot{\theta} + \hat{\phi} r\dot{\phi}\sin\theta$$

وبنفس الطريقة نحصل على التعجيل وكالاتي

$$\vec{a} = \hat{r}(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) + \hat{\theta}(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$$
$$+ \hat{\phi}(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta).$$

تحليل المتجهات

(1) الانحدار Gradient

من الممكن تعريف التحدر (الانحدار) للدالة $u(x,y,z)$ على النحو التالي:

$$\text{grad}(u(x,y,z)) = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}$$

$$du = dx\frac{\partial u}{\partial x} + dy\frac{\partial u}{\partial y} + dz\frac{\partial u}{\partial z}$$

$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

من العلاقات السابقة نلاحظ ما يلي

$$du = d\vec{r}\cdot \text{grad}\,u = |d\vec{r}||\text{grad}\,u|\cos\theta$$

اذا كان اتجاه $\text{grad}\,u$ باتجاه dr

$$\therefore du = |d\vec{r}||\text{grad}\,u|$$

$$|\text{grad}\,u| = \frac{du}{|d\vec{r}|}$$

من الممكن كتابة $\text{grad}\,u$ على النحو التالي

$$\text{grad}\,u = \vec{\nabla} u$$

$$\therefore \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

∇ تسمى المؤثر دلتا

$$\boxed{du = d\vec{r}\cdot\nabla u}$$

(2) الانحراف (تباعد) Divergence

من الممكن تعريف الانحراف للمتجه $\vec{A}$ على النحو التالي

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\text{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

Curl

٣) الالتفاف [الدوران]

يعرف الدوران للمتجه $\vec{A}$ على النحو التالي

$$\text{Curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right] - \hat{y}\left[\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right] + \hat{z}\left[\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right]$$

* مؤثر دلتا ∇ (del) بالإحداثيات الاسطوانية (ρ, ϕ, z)

اذا كانت $u(\rho, \phi, z)$

$$\therefore du = \frac{\partial}{\partial \rho}u\,d\rho + \frac{\partial u}{\partial \phi}d\phi + \frac{\partial u}{\partial z}dz$$

بينا سابقاً ان متجه الموضع $\vec{r}$ في هذه الإحداثيات بالعلاقة التالية

$$\vec{r} = \hat{\rho}\rho + \hat{z}z$$

$$d\vec{r} = \hat{\rho}\,d\rho + \hat{\phi}\,\rho\,d\phi + \hat{z}\,dz$$

$$du = d\vec{r}\cdot\vec{\nabla}u$$

$$\vec{\nabla}u = \hat{\rho}\frac{\partial u}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial \phi} + \hat{z}\frac{\partial u}{\partial z}$$

$$\therefore \boxed{\nabla = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}}$$

مؤثر ∇ في الاحداثيات الاسطوانية

* مؤثر دلتا ∇ في الإحداثيات الكروية

اذا كانت الدالة $u(r, \theta, \phi)$ فان

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi$$

بينا سابقاً ان $\vec{r}$ تعطى بالعلاقة التالية

$$\vec{r} = \hat{r}(\theta,\phi)\,r$$

$$d\vec{r} = \hat{r}\,dr + r\left[\frac{d\hat{r}}{d\theta}d\theta + \frac{d\hat{r}}{d\phi}d\phi\right]$$

$$d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

$$\therefore du = d\vec{r}\cdot\vec{\nabla}u$$

$$\vec{\nabla} u = \hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}}$$

مؤثر ∇ في الاحداثيات الكروية

* الاعتراف للمتجه $\vec{A}$ بالاحداثيات الكروية $\vec{\nabla}\cdot\vec{A}$

$$\vec{A} = \hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi \qquad \vec{A}$$

$$\vec{\nabla}\cdot\vec{A} = \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right]\cdot\left[\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi\right]$$

$$= \underbrace{\hat{r}\cdot\frac{\partial}{\partial r}\vec{A}}_{1} + \underbrace{\frac{\hat{\theta}}{r}\cdot\frac{\partial}{\partial \theta}\vec{A}}_{2} + \underbrace{\frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial}{\partial \phi}\vec{A}}_{3}$$

(1)
$$\frac{\partial \vec{A}}{\partial r} = \frac{\partial}{\partial r}\left[\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi\right]$$

$$= \hat{r}\frac{\partial}{\partial r}A_r + \hat{\theta}\frac{\partial}{\partial r}A_\theta + \hat{\phi}\frac{\partial}{\partial r}A_\phi$$

$$\hat{r}\cdot\frac{\partial \vec{A}}{\partial r} = \frac{\partial}{\partial r}A_r \quad \text{— (1)}$$

(2)
$$\frac{\partial}{\partial \theta}\vec{A} = \frac{\partial}{\partial \theta}\left[\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi\right]$$

$$= \hat{r}\frac{\partial}{\partial \theta}A_r + A_r\frac{\partial \hat{r}}{\partial \theta} + \hat{\theta}\frac{\partial A_\theta}{\partial \theta} + A_\theta\frac{\partial \hat{\theta}}{\partial \theta} + A_\phi\overbrace{\frac{\partial \hat{\phi}}{\partial \theta}}^{=0} + \hat{\phi}\frac{\partial A_\phi}{\partial \theta}$$

$$= \hat{r}\frac{\partial A_r}{\partial \theta} + \hat{\theta}A_r + \hat{\theta}\frac{\partial A_\theta}{\partial \theta} - \hat{r}A_\theta + \hat{\phi}\frac{\partial A_\phi}{\partial \theta}$$

$$\frac{\hat{\theta}}{r}\cdot\frac{\partial}{\partial \theta}\vec{A} = \frac{1}{r}\left[A_r + \frac{\partial A_\theta}{\partial \theta}\right] \quad \text{— (2)}$$

وبنفس الطريقة نحصل على

$$\frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial}{\partial \phi}\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \phi}A_\phi + A_r\sin\theta + A_\theta\cos\theta\right] \quad \text{— (3)}$$

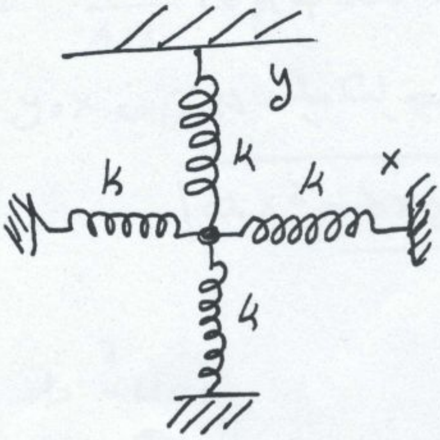
$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial r}A_r + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi + \frac{A_r}{r} + \frac{A_\theta}{r\tan\theta}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial r}A_r + \frac{2A_r}{r} + \frac{1}{r}\frac{\partial}{\partial \theta}A_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi + \frac{A_\theta}{r\tan\theta}}$$

العلاقة الاخيرة يمكن كتابتها بالصيغة التالية

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

المتذبذب التوافقي في بعدين



نفرض ان الجسم تؤثر عليه قوة مركزية [قوة ارجاع التوابض] خطية ويمكن تمثيل القوة F بالعلاقة التالية

$$\vec{F} = -k\vec{r}$$

وتكون معادلات الحركة فيها بالعلاقة التالية

$$m\frac{d^2r}{dt^2} = -k\vec{r}$$

اذا كان الجسم حيث مجموعة من التوابض ومن الشكل نلاحظ ان معادلات الحركة يمكن كتابتها بالصيغة التالية

$$m\ddot{x} = -kx \Longrightarrow m\ddot{x} + kx = 0 \quad \text{—} (1)$$

$$m\ddot{y} = -ky \Longrightarrow m\ddot{y} + ky = 0 \quad \text{—} (2)$$

هاتين المعادلتين التفاضليتين لها الحلول التالية

$$x = A\cos(\omega t + \theta)$$

$$y = B\cos(\omega t + \phi)$$

حيث $\omega = \sqrt{\frac{k}{m}}$ وكل من A و B سعات تحسب من الشروط الابتدائية، كما ان الطور θ و ϕ يحسب من الشروط الابتدائية ايضا.

ولايجاد معادلة المسار نحاول حذف الزمن من هاتين المعادلتين ونحصل على التالي

$$\frac{y}{B} = \cos(\omega t + \phi) = \cos[(\omega t + \theta) + (\phi - \theta)]$$

$$\frac{y}{B} = \cos(\omega t + \theta)\cos(\phi - \theta) - \sin(\omega t + \theta)\sin(\phi - \theta)$$

$$\frac{y}{B} = \frac{x}{A}\cos(\phi - \theta) - \sqrt{1 - \frac{x^2}{A^2}}\,\sin(\phi - \theta)$$

$$\frac{y}{B}-\frac{x}{A}\cos(\phi-\theta)=-\sqrt{1-\frac{x^2}{A^2}}\,\sin(\phi-\theta)$$

نربع طرفي هذه المعادلة

$$\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(\phi-\theta)+\frac{x^2}{A^2}\cos^2(\phi-\theta)=\left(1-\frac{x^2}{A^2}\right)\sin^2(\phi-\theta)$$

$$\frac{x^2}{A^2}-\frac{2xy}{AB}\cos(\phi-\theta)+\frac{y^2}{B^2}=\sin^2(\phi-\theta)$$

هذه المعادلة الأخيرة هي معادلة من الدرجة الثانية للمتغيرين x و y والمعادلة العامة لها هي:

$$ax^2+bxy+cy^2+dx+ey=F$$

هذه المعادلة تمثل الحالات التالية

① قطع ناقص اذا كان b^2-4ac سالب

② قطع مكافئ اذا كان b^2-4ac صفراً

③ قطع زائد اذا كان b^2-4ac موجب

في هذه الحالات التي تحت الدراسة نلاحظ ما يلي

$$b=-\frac{2}{AB}\cos(\phi-\theta),\quad a=\frac{1}{A^2},\quad c=\frac{1}{B^2}$$

$$b^2-4ac=\frac{4}{A^2B^2}\cos^2(\phi-\theta)-\frac{4}{A^2B^2}$$

$$=\frac{4}{A^2B^2}\left[\cos^2(\phi-\theta)-1\right]$$

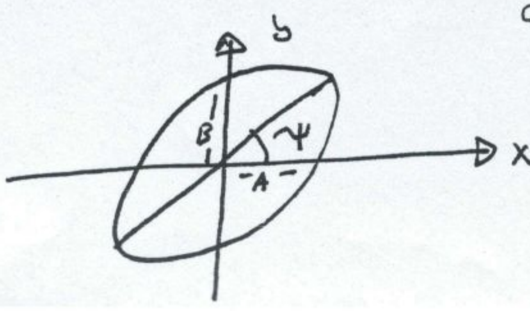
$$=\frac{4}{A^2B^2}\sin^2(\phi-\theta)$$

$$=\left(\frac{2}{AB}\sin(\phi-\theta)\right)^2$$

① $-\left(\frac{2\sin(\phi-\theta)}{AB}\right)^2$

∴ بما ان المقدار سالب فيكون المسار قطع ناقص وكما يلي

* مسار قطع ناقص لمتذبذب توافقي في بعدين



$$\tan 2\psi=\frac{2AB\cos(\phi-\theta)}{A^2-B^2}$$

3)

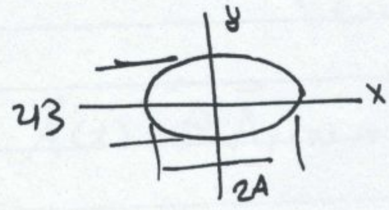
اذا ما قسمنا الى الحالات التالية

① اذا كان فرق الطور $\phi-\theta=\frac{\pi}{2}$

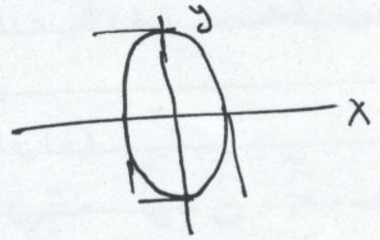
فأن معادلة الحركة تصبح كالتالي

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$$

∴ هذه معادلة قطع ناقص محاوره منطبقة على الاحداثيات x و y وطول كل محور 2A و 2B كما في الشكل



$A>B$



$B>A$

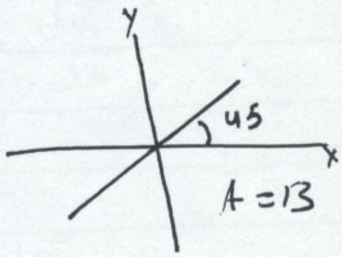
② اذا كان فرق الطور $\phi-\theta=0,\pi,2\pi$

$$\frac{x^2}{A^2}\pm\frac{2xy}{AB}+\frac{y^2}{B^2}=0$$

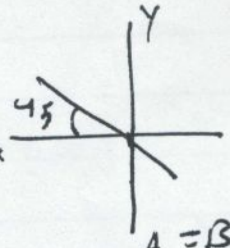
$$\left(\frac{x}{A}\pm\frac{y}{B}\right)^2=0$$

$$\frac{x}{A}\pm\frac{y}{B}=0$$

$$x=\pm\frac{A}{B}y$$ معادلة خط مستقيم



$\phi-\theta=0$



$\phi-\theta=\pi$

س/ اذا كان $B=2A$ جد مقدار الزاوية من الرسم ؟

★مثال حول السرعة والتعجيل في الاحداثيات القطبية

مثال: يتحرك جسيم على مسار حلزوني بحيث موضعه بالاحداثيات القطبية هو كالاتي

$$r = bt^2 \qquad \phi = ct$$

حيث b, c ثوابت، جد السرعة والتعجيل كدالة للزمن t

ج/ من معادلة السرعة في الاحداثيات القطبية

$$\bar{v} = \hat{r} v_r + \hat{\phi} v_\phi$$

$$v_r = \dot{r} = 2bt$$

$$v_\phi = r\dot{\phi} = (bt^2)(c) = bct^2$$

$$\therefore \bar{v} = (2bt)\hat{r} + (bct^2)\hat{\phi}$$

$$\bar{a} = \hat{r}(\ddot{r} - r\dot{\phi}^2) + \hat{\phi}[2\dot{\phi}\dot{r} + r\ddot{\phi}]$$

$$\ddot{r} = 2b \qquad \dot{\phi} = c \qquad \ddot{\phi} = 0$$

$$\dot{r} = 2bt$$

$$\bar{a} = \hat{r}(2b - bt^2c^2) + \hat{\phi}[0 + 2(2bt\,c)]$$

$$= b\hat{r}(2 - t^2c^2) + 4bct\,\hat{\phi}$$

★مسألة حول السرعة والتعجيل في الاحداثيات القطبية

H.W

اذا علمت ان الاحداثيات القطبية لجسم ما

a) $r = b e^{kt}$ $\phi = \omega t$

b) $r = A \cos \omega t$ $\phi = c\omega t$

جد متجه السرعة والتعجيل كدالة للزمن

جد السرعات والتعجيل في الزمن

$t = 0$

<u>اسئلة وحلول الفصل الاول والثاني :</u>

كرة صغيرة تدور بمسار دائري حيث ان متجه الموضع لها

r(t) = ib coswt +j2b sinwt اوجد السرعة والانطلاق والتعجيل والمسافة الى نقطة الاصل والتعجيل عند الزمن

t=0,90 Degrees

الجواب

$$\bar{v}(t) = -\hat{i}b\omega\sin(\omega t) + \hat{j}2b\omega\cos(\omega t)$$

$$|\bar{v}| = \left(b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}} = b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\bar{a}(t) = -\hat{i}b\omega^2\cos\omega t - \hat{j}2b\omega^2\sin\omega t$$

$$|\bar{a}| = b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

$$\text{at} \quad t=0, \quad |\bar{v}| = 2b\omega; \quad \text{at} \quad t=\frac{\pi}{2\omega}, \quad |\bar{v}| = b\omega$$

س١٢\ طائر متجه الموقع له حسب العلاقة التالية

r(t)=ib sinwt +jb coswt + kct2

اثبت ان التعجيل قيمته ثابتة

ج\

$$\bar{v}(t) = \hat{i}b\omega\cos\omega t - \hat{j}b\omega\sin\omega t + \hat{k}2ct$$

$$\bar{a}(t) = -\hat{i}b\omega^2\sin\omega t - \hat{j}b\omega^2\cos\omega t + \hat{k}2c$$

$$|\bar{a}| = \left(b^2\omega^4\sin^2\omega t + b^2\omega^4\cos^2\omega t + 4c^2\right)^{\frac{1}{2}} = \left(b^2\omega^4 + 4c^2\right)^{\frac{1}{2}}$$

س١٣\ حشرة تدور في مسار دائري اوجد السرعة والتعجيل مستخدما الاحداثيات القطبية اذا علمت ان

$r=be^{kt} \quad , \Theta=ct$

ج\١

$$\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta=bke^{kt}\hat{e}_r+bce^{kt}\hat{e}_\theta$$

$$\vec{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta=b\left(k^2-c^2\right)e^{kt}\hat{e}_r+2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2-c^2\right)^2+4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi=\frac{k\left(k^2+c^2\right)}{\left(k^2+c^2\right)^{\frac{1}{2}}\left(k^2+c^2\right)}=\frac{k}{\left(k^2+c^2\right)^{\frac{1}{2}}},\quad \text{a constant}$$

س١٣\ اذا كان متجه الموقع يعطى بالعلاقة التالية H.W

$$r(t)=i\left(1-e^{-it}\right)+je^{kt}$$

س١٤\

An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r=b \qquad \phi=\omega t \qquad \theta=\frac{\pi}{2}\left[1+\frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

استخدم الاحداثيات الكروية لحساب السرعة ؟

$$\bar{v} = \hat{e}_r \dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}$$

$$\bar{v} = \hat{e}_\phi b\omega \sin\left\{\frac{\pi}{2}\left[1+\frac{1}{4}\cos(4\omega t)\right]\right\} - \hat{e}_\theta b\frac{\pi}{2}\omega\sin(4\omega t)$$

$$\bar{v} = \hat{e}_\phi b\omega \cos\left[\frac{\pi}{8}\cos(4\omega t)\right] - \hat{e}_\theta b\omega\frac{\pi}{2}\sin(4\omega t)$$

$$|\bar{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$

س١٥

Given the two vectors **A** = **i** + **j** and **B** = **j** + **k**, find the following:

(a) **A** + **B** and |**A** + **B**|

(b) 3**A** − 2**B**

(c) **A** · **B**

(d) **A** × **B** and |**A** × **B**|

ج١

(a) $\bar{A}+\bar{B} = (\hat{i}+\hat{j}) + (\hat{j}+\hat{k}) = \hat{i} + 2\hat{j} + \hat{k}$

$$|\bar{A}+\bar{B}| = (1+4+1)^{\frac{1}{2}} = \sqrt{6}$$

(b) $3\bar{A} - 2\bar{B} = 3(\hat{i}+\hat{j}) - 2(\hat{j}+\hat{k}) = 3\hat{i} + \hat{j} - 2\hat{k}$

(c) $\bar{A}\cdot\bar{B} = (1)(0) + (1)(1) + (0)(1) = 1$

(d) $$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i}(1-0) + \hat{j}(0-1) + \hat{k}(1-0) = \hat{i} - \hat{j} + \hat{k}$$

$$|\bar{A}\times\bar{B}| = (1+1+1)^{\frac{1}{2}} = \sqrt{3}$$

س١٦

Given the three vectors **A** = 2**i** + **j**, **B** = **i** + **k**, and **C** = 4**j**, find the following:
(a) **A** · (**B** + **C**) and (**A** + **B**) · **C**
(b) **A** · (**B** × **C**) and (**A** × **B**) · **C**
(c) **A** × (**B** × **C**) and (**A** × **B**) × **C**

ج١

(a) $\bar{A}\cdot\left(\bar{B}+\bar{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$\left(\bar{A}+\bar{B}\right)\cdot\bar{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot 4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$

(b) $\bar{A}\cdot\left(\bar{B}\times\bar{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$

$\left(\bar{A}\times\bar{B}\right)\cdot\bar{C}=\bar{A}\cdot\left(\bar{B}\times\bar{C}\right)=-8$

(c) $\bar{A}\times\left(\bar{B}\times\bar{C}\right)=\left(\bar{A}\cdot\bar{C}\right)\bar{B}-\left(\bar{A}\cdot\bar{B}\right)\bar{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

س١٧

Find the angle between the vectors **A** = a**i** + 2a**j** and **B** = a**i** + 2a**j** + 3a**k**. (*Note:* These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, $2a$, and $3a$.)

اوجد قيمة الزاوية بين المتجهين

$$\cos\theta=\frac{\bar{A}\cdot\bar{B}}{AB}=\frac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}}=\frac{5a^2}{a^2\sqrt{5}\sqrt{14}}$$

$$\theta=\cos^{-1}\sqrt{\frac{5}{14}}\approx 53^\circ$$

س١٨

For what value (or values) of q is the vector **A** = **i**q + 3**j** + **k** perpendicular to the vector **B** = **i**q − q**j** + 2**k**?

اوجد قيمة qاذا علمت ان المتجهين متعامدان

ج/

$$0 = \vec{A} \cdot \vec{B} = (q)(q) + (3)(-q) + (1)(2) = q^2 - 3q + 2$$

$$(q-2)(q-1) = 0, \qquad q = 1 \text{ or } 2$$

الفصل الثالث : ديناميك الجسيم

P1

الحركة على خط مستقيم

افترض لدينا كرتين متصادمتين A و B

كما في الشكل أدناه

A → ← B

m_A m_B

$$m_A \frac{\partial \vec{V}_A}{\partial t} = -m_B \frac{\partial \bar{V}_B}{\partial t} \quad \text{—— (1)}$$

ويمكن حساب القوة

$$\bar{F} = m \frac{\partial \bar{V}}{\partial t} \quad \text{—— (2)}$$

$$\bar{F} = \frac{\partial (m\bar{V})}{\partial t} = \frac{\partial \bar{P}}{\partial t} \quad \text{—— (3)}$$

* الزخم الخطي

$$F = \frac{\partial \bar{P}}{\partial t} \quad / \quad \bar{P} = m\bar{V}$$

القوة تساوي المشتقة الزمنية للزخم

$$\frac{d\bar{P}_A}{\partial t} = -\frac{\partial \bar{P}_B}{\partial t} \Rightarrow \frac{\partial}{\partial t}(\bar{P}_A + \bar{P}_B) = 0$$

$$\vec{P}_A + \vec{P}_B = \text{Constant} \quad \text{—— (4)}$$

P2

إذاً

$$\bar{F} = m\frac{\partial^2 \vec{r}}{\partial t^2} = m\bar{a}$$

$$= m\frac{dv}{\partial t} \Rightarrow \frac{d\bar{v}}{dt} = \frac{\bar{F}}{m} = \bar{a}$$

بما أن

$$\bar{a} = \frac{dv}{dt} \longrightarrow \int_{v_0}^{v} dv = \int_0^t a\,dt \longrightarrow (5$$

$$V = at + v_0 = \frac{dx}{dt}$$

$$dx = (at + v_0)\,dt \Rightarrow \int_{x_0}^{x} dx = \int_0^t (at + v_0)\,dt$$

$$X = \tfrac{1}{2}at^2 + v_0 t + X_0 \text{ ——— } (6)$$

حيث V_0 السرعة الابتدائية ، X_0 الموضع الابتدائي

بما أن

$$\bar{F}(x) = m\ddot{X} \quad , \quad \ddot{X} = \frac{d\dot{X}}{dt} = \frac{dX}{dt}\cdot\frac{d\dot{X}}{\partial X}$$

$$\ddot{X} = v\frac{dv}{dX} \text{ ——— } (7)$$

ويمكن كتابة المعادلة السابقة بالشكل

$$F(X) = mv\frac{dv}{dX} = \frac{m}{2}\left(\frac{\partial v^2}{dX}\right) = \frac{dT}{dX} \quad (8$$

حيث ان $T = \frac{1}{2} m v^2$

ومن معادلة 8 $F(x) = \frac{dT}{dx}$

$$\int F(x)\, dx = \int dT \quad \text{—} \; 8^*$$

وبما ان $\int F(x)\, dx$ تمثل الشغل المسلط على الجسم نتيجة تأثير القوة

$$-\frac{dV(x)}{dx} = F(x) \quad \text{—} \; (9$$

حيث ان $V(x)$ تمثل الطاقة الكامنة.

$$\int F(x)\, dx = -\int \frac{dV}{dx}\, dx = -V(x) + C \quad \text{—} \; (10$$

من معادلة 8^* ومعادلة (10)

$$T = -V(x) + C$$

$$T + V(x) = C \quad \text{—} \; (11$$

$$\frac{1}{2} m v^2 + V(x) = C = E \quad \text{—} \; (12)$$

$$\frac{1}{2} m v^2 = E - V(x) \rightarrow v^2 = \frac{2}{m}\left[E - V(x)\right]$$

$$v = \frac{dx}{dt} = \mp \sqrt{\frac{2}{m}\left[E - V(x)\right]} \quad \text{—} \; 13$$

$$t = \int \frac{\mp\, dx}{\sqrt{\frac{2}{m}\left[E - V(x)\right]}} \qquad (14)$$

مثال

أفرض أن قالب قد قذف بسرعة ابتدائية V_0 على سطح مستوي أملس وأن تأثير مقاومة الهواء التي تتناسب مع V أو $F(V) = -cV$ ، أوجد الزمن اللازم لكي يصل القالب الى موضع السكون t ؟

ج/

$$-cV = m\frac{dV}{dt} \quad , \quad \frac{dV}{-cV} = \frac{dt}{m}$$

$$\int_0^t dt = -\int_{V_0}^{V} \frac{m\, dV}{cV} = -\frac{m}{c}\ln\frac{V}{V_0}$$

$$t = -\frac{m}{c}\ln\frac{V}{V_0} \rightarrow \ln\frac{V}{V_0} = -\frac{tc}{m}$$

$$\frac{V}{V_0} = e^{-\frac{tc}{m}} \rightarrow V = V_0\, e^{-\frac{ct}{m}}$$

$$\frac{dx}{dt} = V_0\, e^{-\frac{ct}{m}} \rightarrow x - x_0 = V_0 \int_0^t e^{-\frac{ct}{m}}\, dt$$

P5

$$X - X_0 = -\frac{mV_0}{c} e^{-\frac{ct}{m}} \Big|_0^t$$

$$X - X_0 = -\frac{mV_0}{c}\left(e^{-\frac{ct}{m}} - 1\right)$$

$$X - X_0 = \frac{mV_0}{c}\left(1 - e^{-\frac{ct}{m}}\right)$$

$$X = \frac{mV_0}{c}\left(1 - e^{-\frac{ct}{m}}\right) + X_0$$

مثال : قالب في حالة السكون على سطح افقي املس

في الزمن $t=0$ سلطت عليه قوة افقية متزايدة

$$F = ct$$

جد السرعة والازاحة كدالة للزمن

$$ct = m\frac{dV}{dt} \rightarrow \int_0^t \frac{ct}{m}\, dt = \int_{V_0}^{V} dV$$

$$V - V_0 = \frac{ct^2}{2m} \rightarrow V = \frac{ct^2}{2m}$$

$$\frac{dX}{dt} = \frac{ct^2}{2m} \rightarrow \int_0^X dX = \int_0^t \frac{ct^2}{2m}$$

$$X = \frac{ct^3}{6m}$$

★ مسائل نظرية الزخم

Ex 1) for the a falling body, If we choose the (x) direction tobe positive upward. find

① The equation of total Energy.

② turning points

③ find v from energy equation

④ find (t)

① $E = \frac{1}{2}mv_0^2 + V_0(x) = \frac{1}{2}mv^2 + V(x)$

$V_0(x) \Rightarrow mgx = 0$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$$

② عند نقط الرجوع

$$E = V(x)$$

$v = 0$

$$\frac{1}{2}mv_0^2 = 0 + mgX_{max}$$

نقط ر. $\left\{ X_{max} = \frac{v_0^2}{2g} \right\}$

③ $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgx$ نعوض $\left(\frac{2}{m}\right)$

$$v_0^2 = v^2 + 2gx$$

$$\boxed{v^2 = v_0^2 - 2gx}$$

$$v = (v_0^2 - 2gx)^{1/2}$$

$$\frac{dx}{dt} = (v_0^2 - 2gx)^{1/2}$$

$$\int_0^t dt = \int_0^x \frac{dx}{(v_0^2 - 2gx)^{1/2}}$$

$$t = \frac{1}{2g}\left[(v_0^2 - 2gx)^{1/2}\right]_0^x$$

$$t = \frac{1}{2g}\left[(v_0^2 - 2gx)^{1/2} - (v_0^2)^{1/2}\right]$$

$$t = \frac{1}{2g}\left[(v_0^2 - 2gx)^{1/2} - v_0\right]$$

$v =$

EX-2 | Suppose a block is projected with initial velocity (v_0) on a smooth horizontal plane, but that there is air resistance proportional to (v),

$F(v) = -cv$ where ~~is~~ c is a constant. find

① velocity as a function of (t)

② position as a function of (t)

③ The enough time to move distance (x)

Sol

$$f(v) = mv\frac{dv}{dx} = m\frac{dv}{dt} = -cv$$

$$\int_0^t dt = \int_{v_0}^{v} \frac{m\,dv}{f(v)} = \int_{v_0}^{v} \frac{m\,dv}{-cv}$$

$$t = -\frac{m}{c}\int_{v_0}^{v} \frac{dv}{v} \quad , t = -\frac{m}{c}\ln v\Big|_{v_0}^{v}$$

$$= -\frac{m}{c}(\ln v - \ln v_0) = -\frac{m}{c}\ln\frac{v}{v_0}$$

$$-\frac{c}{m}t = \ln\frac{v}{v_0} \Rightarrow v = v_0 e^{-\frac{c}{m}t}$$

$$\frac{dv}{dt} = v_0 e^{-\frac{c}{m}t}$$

③ $m\frac{dv}{dt} = -cv = mv\frac{dv}{dx}$

$$\frac{dv}{dx} \cdot dx = -c\cancel{v} \rightarrow m\frac{dv}{\ } = -c$$

$$\int_{v_0}^{v} dv = -\frac{c}{m}x \qquad v - v_0 = -\frac{c}{m}x$$

$$v = -\frac{c}{m}x + v_0 = \frac{dx}{dt}$$

$$dt = \frac{dx}{v_0 - \frac{c}{m}x} \Rightarrow t = \int_0^x \frac{dx}{v_0 - \frac{c}{m}x}$$

$$t = -\frac{m}{c}\ln\left(\frac{v_0 - \frac{c}{m}x}{v_0}\right)$$

---

eg-9 The force acting on a particle varies with the distance x according to the power law

$$F(x) = -kx^n$$

① find the potential energy function

② if $v = v_0$ at time $t=0$ and $x=0$, find v as a function of x

③ Determine the turning point of the motion.

Sol.

$$F(x) = -\frac{\partial V(x)}{\partial x}$$

$$\int dV(x) = -\int F(x)dx = \int kx^n dx$$

$$V(x) = \frac{kx^{n+1}}{n+1} + \text{constant}$$

من الشروط الابتدائية

$t=0 \quad , x=0$

↓ صفر

$= \text{Constant} = V(x) \qquad , V(x) = mg\,x$

$c = \text{zero}$

الطاقة الكامنة $V(x) = \dfrac{kx^{n+1}}{n+1}$

② قانون حفظ الطاقة $\frac{1}{2}mv_0^2 + V(x_0) = \frac{1}{2}mv^2 + V(x) = E$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - V(x) \Big] * \frac{2}{m}$$

$$v^2 = v_0^2 - \frac{2}{m}\cdot\frac{kx^{n+1}}{n+1}$$

$$v(x) = \left[v_0^2 - \frac{2kx^{n+1}}{m(n+1)}\right]^{1/2}$$

③ عند نقطة الرجوع الطاقة الكامنة = الطاقة الكلية
والسرعة تساوي صفر

$V(x) = E \qquad v(x) = 0$

$$0 = \sqrt{v_0^2 - \frac{2kx^{n+1}}{m(n+1)}} \quad , v_0^2 = \frac{2kx^{n+1}}{m(n+1)}$$

$$x^{n+1} = \frac{m(n+1)v_0^2}{2k}$$

$$\left[\frac{m(n+1)v_0^2}{\quad}\right]^{\frac{1}{n+2}}$$

eg-15) The force acting on a particle of mass (m) is given by $F = kvx$, k is a constant. The particle passes through the origin with speed v_0 at time $t=0$, find x as a function of (t)

Sol.

$$kvx = mv\frac{dv}{dx}$$

$$m\,dv = kx\,dx \rightarrow \int_{v_0}^{v} dv = \frac{k}{m}\int_0^x x\,dx$$

$$v - v_0 = \frac{kx^2}{2m} \rightarrow v = v_0 + \frac{kx^2}{2m} = \frac{dx}{dt}$$

$$dx = v\,dt \rightarrow dt = \frac{dx}{v}$$

$$\int_0^t dt = \int_0^x \frac{dx}{v_0 + \frac{kx^2}{2m}} \quad , t = \frac{2m}{k}\int_0^x \frac{dx}{\frac{2mv_0}{k} + x^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$t = \frac{2m}{k}\cdot\frac{1}{\sqrt{\frac{2mv_0}{k}}}\tan^{-1}\frac{x}{\sqrt{\frac{2mv_0}{k}}}$$

$$\tan^{-1}\frac{x}{\sqrt{\frac{2mv_0}{k}}} = \frac{kt\sqrt{\frac{2mv_0}{k}}}{2m}$$

$$x = \sqrt{\frac{k}{\ }}\ \tan\left(kt\sqrt{\frac{2mv}{k}}\right.$$

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eg-6) A block slides on a horizontal surface which has been lubricated with a heavy oil such that the block suffers a viscous resistance that varies as the square root of the speed $f(v) = -cv^{1/2}$

initial speed

$v = v_0$, $t = 0$ find v and x as a function of time t

Sol/

$$m\frac{dv}{dt} = -cv^{1/2}$$

$$\int_{v_0}^{v} v^{-1/2}\, dv = -\frac{c}{m}\int_0^t dt$$

$$2v^{1/2}\Big|_{v_0}^{v} = -\frac{c}{m}t$$

$$2v^{1/2} - 2v_0^{1/2} = -\frac{c}{m}t$$

$$2v^{1/2} = -\frac{c}{m}t + 2v_0^{1/2}$$

$$v^{1/2} = -\frac{c}{2m}t + v_0^{1/2}$$

$$v = \left(-\frac{c}{2m}t + v_0^{1/2}\right)^2 = \frac{c^2}{4m^2}t^2 - \frac{c}{m}t v_0^{1/2} + v_0$$

$$\frac{dx}{dt} =$$

EX

The force acting on a particle is $f = a e^{bt}$, find $v(x)$, (x) as a function of time, where and when this particle is stopped? (a, b) is a constant

sol

$$f = m\frac{dv}{dt} \quad , \int_{v_0}^{v} dv = \int_0^t \frac{f\,dt}{m}$$

$$\int_{v_0}^{v} dv = \int_0^t \frac{a e^{bt}}{m} dt \quad , v - v_0 = \frac{a}{mb} e^{bt} \Big|_0^t = \frac{a}{mb}(e^{bt} - 1)$$

$$v = v_0 + \frac{a}{mb}(e^{bt} - 1) = \frac{dx}{dt} \quad , dx = v\,dt$$

$$\int_0^x dx = \int_0^t \left(v_0 + \frac{a}{mb}(e^{bt} - 1)\right) dt$$

$$x = \int_0^t v_0\,dt + \int_0^t \frac{a}{mb} e^{bt} dt - \int_0^t \frac{a}{mb} dt \quad \text{---} (1$$

$$x = v_0 t + \frac{a}{mb^2}(e^{bt} - 1) - \frac{a}{mb} t$$ السؤال كرامة

عندما يتوقف الجسم فإن $v = 0$

$$0 = v_0 + \frac{a}{mb}(e^{bt} - 1)$$

$$\frac{a}{mb} e^{bt} = \left(\frac{a}{mb} - v_0\right)$$

$$e^{bt} = \left(1 - \frac{v_0 mb}{\ }\right) = \frac{a - mb v_0}{\ }$$

2.3 Find the potential energy function $V(x)$ for each of the forces in Problem 2.2.

2.4 A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx$. It is projected from $x = 0$ to the right along the positive x direction with initial kinetic energy $T_0 = 1/2\, kA^2$. k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.5 As in the problem above, the particle is projected to the right with initial kinetic energy T_0 but subject to a force $F(x) = -kx + kx^3/A^2$, where k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.6 A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \alpha/x$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

2.7 A block of mass M has a string of mass m attached to it. A force $\mathbf{F}$ is applied to the string, and it pulls the block up a frictionless plane that is inclined at an angle θ to the horizontal. Find the force that the string exerts on the block.

2.8 Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation

$$\dot{x} = bx^{-3}$$

where b is a positive constant, find the force acting on the particle as a function of x. (*Hint*: $F = m\ddot{x} = m\dot{x}\, d\dot{x}/dx$.)

2.9 A baseball (radius = .0366 m, mass = .145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft). Calculate (**a**) the initial potential energy of the baseball, (**b**) its final kinetic energy, and (**c**) the total energy dissipated by the falling baseball by computing the line integral of the force of air resistance along the baseball's total distance of fall. Compare this last result to the difference between the baseball's initial potential energy and its final kinetic energy. *(Hint: In part (c) make approximations when evaluating the hyperbolic functions obtained in carrying out the line integral.)*

2.10 A block of wood is projected up an inclined plane with initial speed v_0. If the inclination of the plane is 30° and the coefficient of sliding friction $\mu_k = 0.1$, find the total time for the block to return to the point of projection.

2.11 A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the $\frac{3}{2}$ power of the speed:

$$F(v) = -cv^{3/2}$$

If the initial speed of the block is v_0 at $x = 0$, show that the block cannot travel farther than $2mv_0^{1/2}/c$.

2.12 A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed, show that the speed varies with height according to the equations

$$v^2 = Ae^{-2kx} - \frac{g}{k} \quad (\textit{upward motion})$$
$$v^2 = \frac{g}{k} - Be^{2kx} \quad (\textit{downward motion})$$

in which A and B are constants of integration, g is the acceleration of gravity, and $k = c_2/m$ where c_2 is the drag constant and m is the mass of the bullet. (*Note:* x is measured positive upward, and the gravitational force is assumed to be constant.)

2.13 Use the above result to show that, when the bullet hits the ground on its return, the speed is equal to the expression

$$\frac{v_0 v_t}{\left(v_0^2 + v_t^2\right)^{1/2}}$$

in which v_0 is the initial upward speed and

$$v_t = (mg/c_2)^{1/2} = \text{terminal speed} = (g/k)^{1/2}$$

(This result allows one to find the fraction of the initial kinetic energy lost through air friction.)

2.14 A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

2.15 Show that the terminal speed of a falling spherical object is given by

$$v_t = [(mg/c_2) + (c_1/2c_2)^2]^{1/2} - (c_1/2c_2)$$

when *both* the linear and the quadratic terms in the drag force are taken into account.

2.16 Use the above result to calculate the terminal speed of a soap bubble of mass 10^{-7} kg and diameter 10^{-2} m. Compare your value with the value obtained by using Equation 2.4.10.

2.17 Given: The force acting on a particle is the product of a function of the distance and a function of the velocity: $F(x, v) = f(x)g(v)$. Show that the differential equation of motion can be solved by integration. If the force is a product of a function of distance and a function of time, can the equation of motion be solved by simple integration? Can it be solved if the force is a product of a function of time and a function of velocity?

2.18 The force acting on a particle of mass m is given by

$$F = kvx$$

in which k is a positive constant. The particle passes through the origin with speed v_0 at time $t = 0$. Find x as a function of t.

2.19 A surface-going projectile is launched horizontally on the ocean from a stationary warship, with initial speed v_0. Assume that its propulsion system has failed and it is slowed

$$\dot{x}d\dot{x} = \frac{1}{m}F_\circ e^{-cx}dx$$

$$\frac{1}{2}\dot{x}^2 = -\frac{F_\circ}{cm}\left(e^{-cx}-1\right) = \frac{F_\circ}{cm}\left(1-e^{-cx}\right)$$

$$\dot{x} = \left[\frac{2F_\circ}{cm}\left(1-e^{-cx}\right)\right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx} = \dfrac{1}{m}\left(F_\circ \cos cx\right)$

$$\dot{x}d\dot{x} = \frac{F_\circ}{m}\cos cx\, dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{F_\circ}{cm}\sin cx$$

$$\dot{x} = \left(\frac{2F_\circ}{cm}\sin cx\right)^{\frac{1}{2}}$$

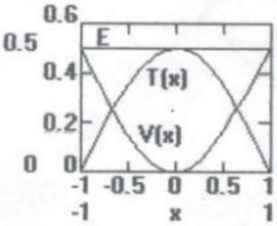
2.3 (a) $V(x) = -\int_{x_\circ}^{x}\left(F_\circ + cx\right)dx = -F_\circ x - \dfrac{cx^2}{2} + C$

(b) $V(x) = -\int_{x_\circ}^{x} F_\circ e^{-cx}dx = \dfrac{F_\circ}{c}e^{-cx} + C$

(c) $V(x) = -\int_{x_\circ}^{x} F_\circ \cos cx\, dx = -\dfrac{F_\circ}{c}\sin cx + C$

2.4 (a) $F(x) = -\dfrac{dV(x)}{dx} = -kx$

$$V(x) = \int_0^x kx\, dx = \frac{1}{2}kx^2$$



(b) $T_\circ = T(x) + V(x)$

$$T(x) = T_\circ - V(x) = \frac{1}{2}k\left(A - x^2\right)$$

(c) $E = T_\circ = \dfrac{1}{2}kA^2$

(d) *turning points* @ $T(x_1) \to 0 \quad \therefore x_1 = \pm A$

2.5 (a) $F(x) = -kx + \dfrac{kx^3}{A^2}$ so $V(x) = \int_0^x \left(kx - \dfrac{kx^3}{A^2}\right)dx = \dfrac{1}{2}kx^2 - \dfrac{1}{4}\dfrac{kx^4}{A^2}$

(b) $T(x) = T_\circ - V(x) = T_\circ - \dfrac{1}{2}kx^2 + \dfrac{1}{4}\dfrac{kx^4}{A^2}$

(c) $E = T_\circ$

(d) $V(x)$ has maximum at $|F(x_m)| \to 0$

$$kx_m - \frac{kx_m^{\ 3}}{A^2} = 0 \qquad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \to 0$ let $u = x_1^{\ 2}$

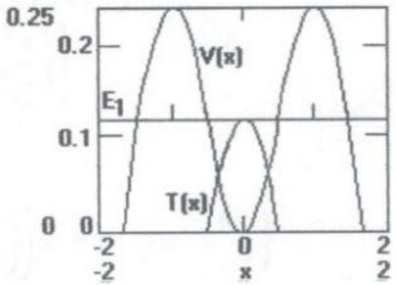
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u, we obtain

$$u = A^2\left[1 \pm \left(1 - \frac{4E}{kA^2}\right)^{\frac{1}{2}}\right]$$

or

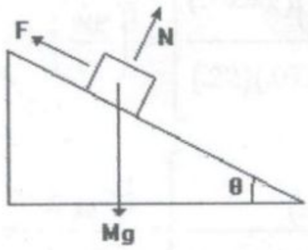
$$x_1 = \pm A\left[1 - \sqrt{\left(1 - \frac{4E}{kA^2}\right)}\right]^{\frac{1}{2}}$$



2.6 $\dot{x} = v(x) = \frac{\alpha}{x} \qquad \ddot{x} = -\frac{\alpha}{x^2}\dot{x} = -\frac{\alpha^2}{x^3}$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$$F \geq Mg\sin\theta$$

2.8 $F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m\left(bx^{-3}\right)\left(-3bx^{-4}\right)$$

$$F = -3mb^2x^{-7}$$

2.9 (a) $V = mgx = (.145kg)\left(9.8\frac{m}{s^2}\right)(1250\,ft)\left(.3048\frac{m}{ft}\right) = 541J$

$$x=\frac{F_\circ}{2m}t_1^{\,2}+\frac{F_\circ}{m}t_1\left(t-t_1\right)+\frac{1}{2}\frac{2F_\circ}{m}\left(t-t_1\right)^2$$

At $\quad t=2t_1$: $\quad x=\frac{F_\circ}{2m}t_1^{\,2}+\frac{F_\circ}{m}t_1^{\,2}+\frac{F_\circ}{m}t_1^{\,2}=\frac{5F_\circ}{2m}t_1^{\,2}$

2.11 $\quad a=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\cdot\frac{dv}{dx}=-\frac{c}{m}v^{\frac{3}{2}}$

$$v^{-\frac{1}{2}}dv=-\frac{c}{m}dx$$

$$\int_{v_\circ}^{\circ}v^{-\frac{1}{2}}dv=\int_0^{x_{max}}-\frac{c}{m}dx$$

$$-2v_\circ^{\frac{1}{2}}=-\frac{c}{m}x_{max}$$

$$x_{max}=\frac{2mv_\circ^{\frac{1}{2}}}{c}$$

1.5

$$\int_{v_0}^{0}v^{1/2}\,dv=\int_0^{x_{max}}-\frac{c}{m}dx$$

$$\frac{2v^{3/2}}{3}\Big|_{v_0}^{0}=-\frac{c}{m}x_{max}$$

$$-\frac{2}{3}v_0^{3/2}=-\frac{c}{m}x_{max}$$

$$x_{max}=\frac{2m}{3c}v_0^{3/2}$$

2.12 $\quad$ Going up: $\quad F_x=-mg\sin30^\circ-\mu mg\cos30^\circ$

$$\ddot{x}=-g\left(\sin30^\circ+0.1\cos30^\circ\right)=-5.749\frac{m}{s^2}$$

$$v=v_\circ+at$$

at the highest point $v=0$ so $t_{up}=-\frac{v_\circ}{a}=0.174v_\circ s$

$$x_{up}=v_\circ t_{up}+\frac{1}{2}at_{up}^2=0.174v_\circ^{\,2}-.087v_\circ^{\,2}=0.087v_\circ^{\,2}m$$

Going down: $\quad x_\circ'=0.087v_\circ^{\,2}$, $\quad v_\circ'=0$, $\quad a'=-9.8\left(0.5-0.0866\right)$

$$x_{down}=0=0.087v_\circ^{\,2}-\frac{1}{2}4.0513t_{down}^2$$

$$t_{down}=0.207v_\circ s$$

$$t_{total}=t_{up}+t_{down}=0.381v_\circ s$$

2.13 $\quad$ At the top $v=0$ so $\quad e^{-2kx_{max}}=\dfrac{\frac{g}{k}}{\frac{g}{k}+v_\circ^{\,2}}$

Coming down $x_\circ=x_{max}$ and at the bottom $x=0$

$$v^2=\frac{g}{k}-\left(\frac{g}{k}\right)^2\frac{1}{\left(\frac{g}{k}+v_\circ^{\,2}\right)}(1)=\frac{\left(\frac{g}{k}\right)v_\circ^{\,2}}{\frac{g}{k}+v_\circ^{\,2}}$$

$$v=\frac{v_t v_\circ}{\left(v_t^{\,2}+v_\circ^{\,2}\right)^{\frac{1}{2}}},\quad v_t=\sqrt{\frac{g}{k}}=\sqrt{\frac{mg}{c_2}}$$

2.14 Going up: $F_x=-mg-c_2v^2$

$$a=v\frac{dv}{dx}=-g-kv^2,\quad k=\frac{c^2}{m}$$

$$\int_{v_\circ}^{v}\frac{vdv}{-g-kv^2}=\int_0^x dx$$

$$-\frac{1}{2k}\ln\left(-g-kv^2\right)\Big|_{v_\circ}^{v}=x$$

$$\frac{g+kv^2}{g+kv_\circ^{\,2}}=e^{-2kx}$$

$$v^2=\left(\frac{g}{k}+v_\circ^{\,2}\right)e^{-2kx}-\frac{g}{k}$$

Going down: $F_x=-mg+c_2v^2$

$$v\frac{dv}{dx}=-g+kv^2$$

$$\int_0^{v}\frac{vdv}{-g+kv^2}=\int_0^x dx$$

$$\frac{1}{2k}\ln\left(-g+kv^2\right)\Big|_0^{v}=x-x_\circ$$

$$1-\frac{k}{g}v^2=e^{2kx}e^{-2kx_\circ}$$

$$v^2=\frac{g}{k}-\left(\frac{g}{k}e^{-2kx_\circ}\right)e^{2kx}$$

2.15 $$m\frac{dv}{dt}=mg-c_1v-c_2v^2$$

$$\int_0^t\frac{dt}{m}=\int_0^v\frac{dv}{mg-c_1v-c_2v^2}$$

Using $\displaystyle\int\frac{dx}{a+bx+cx^2}=\frac{1}{\sqrt{b^2-4ac}}\ln\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}$,

$$\frac{t}{m}=\frac{1}{\sqrt{c_1^{\,2}+4mgc_2}}\ln\frac{-2c_2v-c_1-\sqrt{c_1^{\,2}+4mgc_2}}{-2c_2v-c_1+\sqrt{c_1^{\,2}+4mgc_2}}\Bigg|_0^v$$

$$\frac{t}{m}\left(c_1^{\,2}+4mgc_2\right)^{\frac{1}{2}}=\ln\frac{\left(2c_2v+c_1+\sqrt{c_1^{\,2}+4mgc_2}\right)\left(c_1-\sqrt{c_1^{\,2}+4mgc_2}\right)}{\left(2c_2v+c_1-\sqrt{c_1^{\,2}+4mgc_2}\right)\left(c_1+\sqrt{c_1^{\,2}+4mgc_2}\right)}$$

as $t\to\infty$, $2c_2v_t+c_1-\sqrt{c_1^{\,2}+4mgc_2}=0$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

Alternatively, when $v=v_t$,

$$m\frac{dv}{dt}=0=mg-c_1v_t-c_2v_t^{\,2}$$

$$v_t=-\frac{c_1}{2c_2}+\left[\left(\frac{c_1}{2c_2}\right)^2+\frac{mg}{c_2}\right]^{\frac{1}{2}}$$

2.16 $a=v\dfrac{dv}{dx}=-\dfrac{k}{m}x^{-2}$

$$\int_0^v v\,dv=\int_b^x-\frac{k\,dx}{mx^2}$$

$$\frac{1}{2}v^2=\frac{k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

$$v=\frac{dx}{dt}=\left[\frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)\right]^{\frac{1}{2}}=\left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt=\int_b^0\left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}}dx=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_1^0\left(\frac{\frac{x}{b}}{1-\frac{x}{b}}\right)^{\frac{1}{2}}d\left(\frac{x}{b}\right)$$

Since $x\le b$, say $\dfrac{x}{b}=\sin^2\theta$

$$t=\left(\frac{mb^3}{2k}\right)^{\frac{1}{2}}\int_{\frac{\pi}{2}}^0\frac{\sin\theta\left(2\sin\theta\cos\theta\,d\theta\right)}{\cos\theta}=\left(\frac{2mb^3}{k}\right)^{\frac{1}{2}}\int_{\frac{\pi}{2}}^0\sin^2\theta\,d\theta$$

$$t=\left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}\pi$$

★مسائل وحلول

Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions:

(**a**) $F_x = F_0 + ct$

(**b**) $F_x = F_0 \sin ct$

(**c**) $F_x = F_0 e^{ct}$

where F_0 and c are positive constants.

(a) $\ddot{x} = \frac{1}{m}(F_\circ + ct)$

$$\dot{x} = \int_0^t \frac{1}{m}(F_\circ + ct)\,dt = \frac{F_\circ}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_\circ}{m}t + \frac{c}{2m}t^2 \right) dt = \frac{F_\circ}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_\circ}{m}\sin ct$

$$\dot{x} = \int_0^t \frac{F_\circ}{m}\sin ct\,dt = -\frac{F_\circ}{cm}\cos ct\Big|_0^t = \frac{F_\circ}{cm}(1 - \cos ct)$$

$$x = \int_0^t \frac{F_\circ}{cm}(1 - \cos ct)\,dt = \frac{F_\circ}{cm}\left(1 - \frac{1}{c}\sin ct\right)$$

(c) $\ddot{x} = \frac{F_\circ}{m}e^{ct}$

$$\dot{x} = \frac{F_\circ}{cm}e^{ct}\Big|_0^t = \frac{F_\circ}{cm}(e^{ct} - 1)$$

$$x = \frac{F_\circ}{cm}\left(\frac{1}{c}e^{ct} - \frac{1}{c} - t\right) = \frac{F_\circ}{c^2 m}(e^{ct} - 1 - ct)$$

Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force functions:

(a) $F_x = F_0 + cx$

(b) $F_x = F_0 e^{-cx}$

(c) $F_x = F_0 \cos cx$

where F_0 and c are positive constants.

(a) $$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x}\,\frac{d\dot{x}}{dx}$$

$$\dot{x}\,\frac{d\dot{x}}{dx} = \frac{1}{m}\left(F_\circ + cx\right)$$

$$\dot{x}d\dot{x} = \frac{1}{m}\left(F_\circ + cx\right)dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{1}{m}\left(F_\circ x + \frac{cx^2}{2}\right)$$

$$\dot{x} = \left[\frac{x}{m}\left(2F_\circ + cx\right)\right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx} = \dfrac{1}{m}F_{\circ}e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m}F_{\circ}e^{-cx}dx$$

$$\frac{1}{2}\dot{x}^2 = -\frac{F_{\circ}}{cm}\left(e^{-cx}-1\right) = \frac{F_{\circ}}{cm}\left(1-e^{-cx}\right)$$

$$\dot{x} = \left[\frac{2F_{\circ}}{cm}\left(1-e^{-cx}\right)\right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}\left(F_{\circ}\cos cx\right)$

$$\dot{x}d\dot{x} = \frac{F_{\circ}}{m}\cos cx\, dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{F_{\circ}}{cm}\sin cx$$

$$\dot{x} = \left(\frac{2F_{\circ}}{cm}\sin cx\right)^{\frac{1}{2}}$$

Find the potential energy function V(x) for each of the forces in Problem for Q1 and Q2

(a) $V(x) = -\int_{x_\circ}^{x} (F_\circ + cx)\,dx = -F_\circ x - \frac{cx^2}{2} + C$

(b) $V(x) = -\int_{x_\circ}^{x} F_\circ e^{-cx}\,dx = \frac{F_\circ}{c} e^{-cx} + C$

(c) $V(x) = -\int_{x_\circ}^{x} F_\circ \cos cx\,dx = -\frac{F_\circ}{c} \sin cx + C$

A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \alpha/x$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

$$\dot{x} = v(x) = \frac{\alpha}{x} \qquad \ddot{x} = -\frac{\alpha}{x^2}\dot{x} = -\frac{\alpha^2}{x^3}$$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation

$$\dot{x} = bx^{-3}$$

$$F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m\left(bx^{-3}\right)\left(-3bx^{-4}\right)$$

$$F = -3mb^{2}x^{-7}$$

A baseball (radius = .0366 m, mass = .145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft). Calculate (**a**) the initial potential energy of the baseball,

$$V = mgx = (.145kg)\left(9.8\frac{m}{s^2}\right)(1250\,ft)\left(.3048\frac{m}{ft}\right) = 541J$$

A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the $\frac{3}{2}$ power of the speed:

$$F(v) = -cv^{3/2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} = -\frac{c}{m} v^{\frac{3}{2}}$$

$$v^{-\frac{1}{2}} dv = -\frac{c}{m} dx$$

$$\int_{v_o}^{v} v^{-\frac{1}{2}} dv = \int_0^{x_{max}} -\frac{c}{m} dx$$

$$-2v_o^{\frac{1}{2}} = -\frac{c}{m} x_{max}$$

$$x_{max} = \frac{2mv_o^{\frac{1}{2}}}{c}$$

A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

5 $$a = v\frac{dv}{dx} = -\frac{k}{m}x^{-2}$$

$$\int_0^v v\,dv = \int_b^x -\frac{k\,dx}{mx^2}$$

$$\frac{1}{2}v^2 = \frac{k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$v = \frac{dx}{dt} = \left[\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)\right]^{\frac{1}{2}} = \left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt = \int_b^0 \left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}} dx = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_1^0 \left(\frac{\frac{x}{b}}{1-\frac{x}{b}}\right)^{\frac{1}{2}} d\left(\frac{x}{b}\right)$$

ce $x \le b$, say $\frac{x}{b} = \sin^2\theta$

$$t = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \frac{\sin\theta\,(2\sin\theta\cos\theta\,d\theta)}{\cos\theta} = \left(\frac{2mb^3}{k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \sin^2\theta\,d\theta$$

$$t = \left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}\pi$$

The force acting on a particle of mass m is given by

$$F = kvx$$

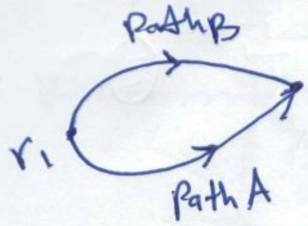
in which k is a positive constant. The particle passes through the origin with speed v_0 at time $t = 0$. Find x as a function of t.

القوة المحافظة
(Conservation)

★(تابع للفصل الثالث القوة المحافظة)

The Conditions of conservation force :-

① if the work is not depended for path the force is conservation and its field called conservation field.



* W_A الشغل اللازم لنقل جسم من r_1 الى r_0 على المسار A

$$W_A = \int_{r_1}^{r_0} \bar{F} \cdot d\bar{r}$$

$$W_{path A} = W_{path B}$$

$$\int_{r_1}^{r_0} \bar{F} \cdot d\bar{r} = \int_{r_1}^{r_0} \bar{F} \cdot d\bar{r}$$

$$\int_{r_1}^{r_0} \bar{F} \cdot d\bar{r} - \int_{r_1}^{r_0} \bar{F} \cdot d\bar{r} = 0 \rightarrow \int_{r_1}^{r_0} \bar{F} \cdot d\bar{r} + \int_{r_0}^{r_1} \bar{F} \cdot d\bar{r} = 0$$

② $$\oint \bar{F} \cdot d\bar{r} = 0$$

الشرط الثاني من شروط القوة المحافظة

اذا كان التكامل المغلق للقوة حول مسار مغلق يساوي صفر تكون القوة محافظة.

(3) if $\bar{F} = -\nabla V$

إذا كانت القوة تساوي سالب انحدار الجهد تكون القوة محافظة.

(4) if $\nabla \times \bar{f} = 0$ then the force is conservation

(5) if $E = K.E + P.E = \text{constant}$

then the force is conservation.

Ex $F = \hat{i}(y^2 z^3 - 6xz^2) + \hat{j}(2xyz^3) + \hat{k}(3xy^2z^2 - 6x^2z)$

show that if the force is conservation or no?

$$\bar{\nabla} \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\bar{\nabla} \times \bar{f} = \hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{j}\left(\frac{\partial f_z}{\partial x} - \frac{\partial}{\partial z} f_x\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial}{\partial y} f_x\right)$$

$$= \hat{i}(6xyz^2 - 6xyz^2) + \hat{j}(3y^2z^2 - 12xz) - (3y^2z^3 - 12xz)]$$

$$+ \hat{k}(2y\hat{z} - 2yz^3) = \text{Zero}.$$

Ex $V = x^2 + xy + xz$ find the force field?

$$\bar{F} = -\nabla V = -\hat{i}\frac{\partial V}{\partial x} - \hat{j}\frac{\partial V}{\partial y} - \hat{k}\frac{\partial V}{\partial z}$$

$$\bar{F} = -\hat{i}(2x + y + z) - \hat{j}(x) - \hat{k}(x)$$

Ex Show that if the force is conservation or no?

$$\bar{F} = \hat{i}xy + \hat{j}xz + \hat{k}yz$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix} = \hat{i}(z-x) - \hat{j}(0-0) + \hat{k}(z-x)$$

$$= \hat{i}(z-x) + \hat{k}(z-x) \neq 0$$

التدوير لا يساوي صفر ، فالقوة غير محافظة

Ex when the force field is conservation?

$$\bar{F} = \hat{i}(ax+by) + \hat{j}cxy$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+by^2 & cxy & 0 \end{vmatrix} = \hat{k}(c-2b)y$$

if the force is conservation then $\nabla \times \bar{f} = 0$

$c - 2b = 0 \Rightarrow c = 2b$

تتميز القوى المركزية من الحالات المهمة للقوة المحافظة، وحدود القوة المركزية تكتب بالمحاور الكروية بالشكل التالي

$$\vec{F} = F(r)\,\hat{r}$$

$$\vec{r} = \hat{r}\,r$$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$\boxed{\vec{F} = \frac{F(r)}{r}\,\vec{r}}$$

"تمارين الفصل الثاني"

مثال / لدينا القوة التي مركباتها تعطى بالعلاقات التالية

$$F_x = axy \;,\; F_y = -az^2 \;,\; F_z = -ax^2$$

حيث a مقدار ثابت. هل هذه القوة محافظة ام لا؟

الجواب / لا

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy & -az^2 & -ax^2 \end{vmatrix}$$

$$= \hat{x}(2az) + \hat{y}(2ax) - \hat{z}(ax) \neq 0$$

∴ القوة غير محافظة.

س/ جد قوة المجال $\vec{F}$ لدالة الجهد التالية

$U = x^2 + xy + xz = V(x, y, z)$. ثم برهن ان القوة محافظة؟

الحل / باستخدام العلاقة

① $$\vec{F} = -\nabla V$$

$$= -\hat{x}(2x+y+z) - \hat{y}x - \hat{z}x$$

② $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -(2x+y+z) & -x & -x \end{vmatrix} = \hat{x}(0) - \hat{y}(-1+1) + \hat{z}(-1+1) = 0$$

ميكانيك تحليلي

امثله + مسائل محلوله

كورس اول

س٣/ هل القوة التالية محافظة - ام لا ؟

$$\vec{F} = \hat{x}\,xy + \hat{y}\,xz + \hat{z}\,yz$$

الجواب / غير محافظة

$$\vec{\nabla} \times \vec{F} = \hat{x}(z-x) + \hat{z}(z-x)$$

س٤/ ما قيمة الثوابت a,b,c والتي تكون فيها القوة

$\vec{F} = \hat{x}(ax+by^2) + \hat{y}(cxy)$ محافظة ؟

الجواب

$$\vec{\nabla} \times \vec{F} = \hat{z}[c-2b]y$$

$$\therefore \vec{\nabla} \times \vec{F} = 0 \quad \text{if } c = 2b$$

$\therefore$ a لها اي قيمة

س/ جسم كتلته m يتحرك في المستوي x y وحركته الموضح له يعطى بالعلاقة التالية

$$\vec{r} = \hat{x}\,a\cos\omega t + \hat{y}\,b\sin\omega t$$

حيث ان a, b ثوابت موجبة و a > b و ω سرعة زاوية

١- برهن ان مسار الجسم له شكل قطع ناقص

٢- برهن ان القوة المؤثرة على الجسم تتجه دائماً باتجاه نقطة الاصل.

٣- برهن ان القوة المؤثرة على الجسم هي محافظة.

٤- جد الطاقة الكامنة V

٥- جد الشغل المنجز او اللازم لتحريك الجسم من النقطة A الى النقطة B.

٦- جد معادلة الطاقة الحركية للجسم

٧- جد الطاقة الحركية للنقطة A وكذلك في B

٨- برهن ان الشغل المنجز من النقطة A الى النقطة B ما هو الا الفرق في الطاقة الحركية للجسم في النقطة A و B

٩- جد الطاقة الكامنة في النقطة A وفي النقطة B

١٠- برهن ان الطاقة الكلية تساوي مقدار ثابت

١١- برهن ان الشغل المنجز لاكمال دورة واحدة حول القطع الناقص يساوي صفر.

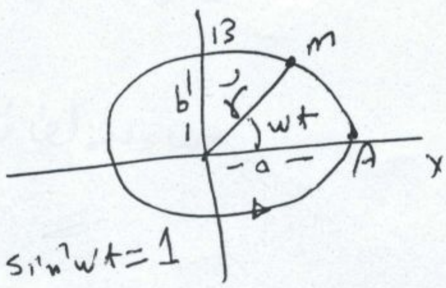
الحل/

(1) $\vec{r} = \hat{x}\, a\cos\omega t + \hat{y}\, b\sin\omega t$

$r = \hat{x}\, x + \hat{y}\, y$

$x = a\cos\omega t$

$y = b\sin\omega t$



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2\omega t + \sin^2\omega t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$ قطع ناقص

نصف القطر الكبير (a) متطابق مع المحور x

= = الصغير (b) = y

(2) $\vec{F} = m\vec{a}$

$$= m\frac{d^2r}{dt^2} = m\left[-\hat{x}\, a\omega^2\cos\omega t - \hat{y}\, b\omega^2\sin\omega t\right]$$

$$= -m\omega^2\left[\hat{x}\, a\cos\omega t + \hat{y}\, b\sin\omega t\right]$$

$$\boxed{\vec{F} = -m\omega^2\vec{r}}$$

من المعادلة الاخيرة ووجود الاشارة السالبة نلاحظ ان القوة باتجاه معاكس ل r اي متجهة الى نقطة الاصل.

(3) $F = -m\omega^2\left[\hat{x}\, x + \hat{y}\, y\right]$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -m\omega^2 x & -m\omega^2 y & 0 \end{vmatrix} = \hat{x}[0] - \hat{y}[0] + \hat{z}[0]$$

∴ القوة محافظة

(4) $\vec{F} = -\nabla V$

$F = -m\omega^2\left[\hat{x}\, x + \hat{y}\, y\right]$

$$-\nabla V = -\left[\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}\right]$$

$$-m\omega^2\left[\hat{x}\, x + \hat{y}\, y\right] = -\left[\hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}\right]$$

$$\frac{\partial V}{\partial x} = m\omega^2 x \quad , \quad \frac{\partial V}{\partial y} = m\omega^2 y \quad , \quad \frac{\partial V}{\partial z} = 0$$

$$\int dV = m\omega^2\int x\,dx \quad , \quad \int dV = \int m\omega^2 y\,dy$$

$V=\frac{1}{2}m w^2 x^2$

$V=\frac{1}{2}m w^2 y^2$ } $V=\frac{1}{2}m w^2[x^2+y^2]$

$$V=\frac{1}{2}m w^2 r^2$$ معادلة الطاقة الكامنة

(5) الطريقة الأولى

$$W_{AB}=\int_a^b \vec{F}\cdot d\vec{r}=\int_a^b(-mw^2 r)\cdot dr=-\int_a^b m w^2 r\cdot dr$$

$$=-\frac{1}{2}m w^2\int_a^b d(r\cdot r)$$

$$=-\frac{1}{2}m w^2\int_a^b d(r^2)$$

$$=-\frac{1}{2}m w^2 r^2\Big|_a^b=\frac{1}{2}m w^2[a^2-b^2]$$

الطريقة الثانية

$$W_{AB}=\int_a^b \vec{F}\cdot d\vec{r}$$

$\vec{r}=\hat{x}x+\hat{y}y$

$dr=\hat{x}\,dx+\hat{y}\,dy$

$x=a\cos wt \quad , dx=-aw\sin wt$

$y=b\sin wt \quad , dy=bw\cos wt$

$wt=\frac{\pi}{2}$
$t=\frac{\pi}{2w}$

$dr=[-\hat{x}aw\sin wt+\hat{y}bw\cos wt]\,dt$

$F=-mw^2[\hat{x}a\cos wt+\hat{y}b\sin wt]$

$$=\int_0^{\frac{\pi}{2w}}\underbrace{[-mw^2[\hat{x}a\cos wt+\hat{y}b\sin wt]}_{F}\cdot\underbrace{[-\hat{x}aw\sin wt+\hat{y}bw\cos wt]}_{dr}\,dt$$

A عند $wt=0 \Rightarrow t=0$

B عند $wt=\frac{\pi}{2}\Rightarrow t=\frac{\pi}{2w}$

$$=\int_0^{\frac{\pi}{2w}}[mw^3a^2\cos wt\sin wt-mw^3b^2\sin wt\cos wt]\,dt$$

$$= \int_0^{\frac{\pi}{2w}} m w^3 (a^2 - b^2) \cos wt \sin wt \, dt$$

$$= \frac{1}{2w} m w^3 (a^2 - b^2) \sin^2 wt \Big|_0^{\frac{\pi}{2w}}$$

$$= \frac{1}{2} m w^2 (a^2 - b^2)$$

(6) $\vec{r} = \hat{x} x + \hat{y} y = \hat{x} a \cos wt + \hat{y} b \sin wt$

$$\vec{V} = \frac{d\vec{r}}{dt} = -\hat{x} a w \sin wt + \hat{y} b w \cos wt$$

$$V^2 = \vec{V} \cdot \vec{V} = a^2 w^2 \sin^2 wt + b^2 w^2 \cos^2 wt$$

$$T = \frac{1}{2} m V^2 = \frac{1}{2} m w^2 [a^2 \sin^2 wt + b^2 \cos^2 wt]$$

(7) $T_A \Rightarrow \quad wt = 0$ ع النقطة A

$$\boxed{T_A = \frac{1}{2} m w^2 b^2}$$

$T_B \Rightarrow \quad wt = \frac{\pi}{2}$ ع النقطة B

$$\boxed{T_B = \frac{1}{2} m w^2 a^2}$$

(8)

$W_{AB} = T_B - T_A$ الشغل المبذول هو الفرق بين الطاقتين الحركية

(9) $V = \frac{1}{2} m w^2 r^2$

at $A \Rightarrow r = a$

$$\boxed{V_A = \frac{1}{2} m w^2 a^2}$$

at $B \Rightarrow r = b$

$$\boxed{V_B = \frac{1}{2} m w^2 b^2}$$

(10)

$$E = T + V$$

$$V = \frac{1}{2} m w^2 r^2 = \frac{1}{2} m w^2 [x^2 + y^2] = \frac{1}{2} m w^2 (a^2 + b^2)$$

$$E = \frac{1}{2} m w^2 (a^2 + b^2)$$

$$E = const.$$

at $A \Rightarrow \quad E = T_A + V_A = \frac{1}{2} m w^2 (b^2 + a^2)$

at $B \Rightarrow \quad E = T_B + V_B = \frac{1}{2} m w^2 (b^2 + a^2)$

∴ الطاقة محفوظة

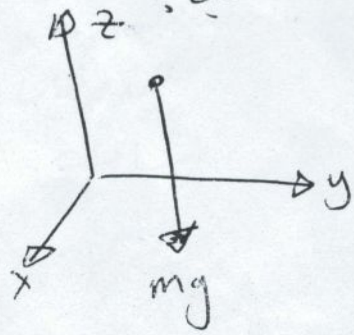
$\boxed{wt = 2\pi}$

(11)

$$W_{A \to B} = \int_0^{\frac{2\pi}{w}} \vec{F} \cdot d\vec{r} = \int_0^{\frac{2\pi}{w}} m w^2 (a^2 - b^2) \sin wt \cos wt \, dt$$

$$= \frac{1}{2w} m w^3 [a^2 - b^2] \sin^2 wt \Big|_0^{\frac{2\pi}{w}} = 0$$

س/ جسم كتلته m سقط سقوطاً حراً شاقولياً باتجاه المحور Z (اهمل مقاومة الهواء) جد مركبة القوة المؤثرة على الجسم ومعادلة الطاقة الكامنة ومعادلة الطاقة الكلية؟



∵ قوة الجاذبية قوة محافظة ∴ فإن

$$F_x = -\frac{\partial V}{\partial x} = 0$$

$$F_y = -\frac{\partial V}{\partial y} = 0$$

$$F_z = -\frac{\partial V}{\partial z} = -mg$$

$$V = mgz + const.$$

$$T = \frac{1}{2} m \dot{z}^2$$

$$E = V + T = mgz + \frac{1}{2} m \dot{z}^2 + const.$$

س٩ / جسم كتلته m يتحرك حركة توافقية متجانسة في ثلاث اتجاهات
جد مركبات القوة، الطاقة الكامنة، الطاقة الكلية، الطاقة الحركية ؟
الحل /

$$F_x = -\frac{\partial V}{\partial x}, \quad F_x = -kx = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y}, \quad F_y = -ky = -\frac{\partial V}{\partial y}$$
$$F_z = -\frac{\partial V}{\partial z}, \quad F_z = -kz = -\frac{\partial V}{\partial z} \quad \Rightarrow$$

$$V(x,y,z) = \frac{1}{2}k(x^2+y^2+z^2) + const.$$
$$= \frac{1}{2}kr^2$$
$$T = \frac{1}{2}mv^2$$
$$E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kr^2 + const.$$

س١٠ / لدينا جسم كتلته M تؤثر عليه دالة الجهد التالية

$$U(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{k}{r}$$

جد مركبات القوة المؤثرة عليه F_x, F_y, F_z ثم برهن ان القوة المؤثرة عليه تساوي $F = -\frac{k}{r^2}\hat{r}$

الحل /

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

$$\frac{\partial V}{\partial x} = \frac{1}{2}\frac{k(2x)}{(x^2+y^2+z^2)^{3/2}} = \frac{kx}{r^3}$$

$$F_x = -\frac{kx}{r^3}, \quad F_y = -\frac{ky}{r^3}, \quad F_z = -\frac{kz}{r^3}$$

$$F = \hat{x}F_x + \hat{y}F_y + \hat{z}F_z$$
$$= -\frac{k}{r^3}[\hat{x}x + \hat{y}y + \hat{z}z]$$
$$= -\frac{k}{r^3}\vec{r} \Rightarrow \vec{r} = \hat{r}r \Rightarrow \boxed{\vec{F} = -\frac{k}{r^2}\hat{r}}$$

س/ جسم يتحرك في ثلاث ابعاد وحركة قدرها v وتحت تأثير قوة محافظة $\vec{F}$ استخدم قانون نيوتن الثاني في الحركة ثم وضح رياضياً ان مجموع الطاقة الحركية والطاقة الكامنة تساوي مقدار ثابت.

الحل/

$$T+V=\text{const.}$$

$$\vec{v}=\hat{x}\dot{x}+\hat{y}\dot{y}+\hat{z}\dot{z}$$

$$\vec{F}=\hat{x}F_x+\hat{y}F_y+\hat{z}F_z$$

$$F=m\vec{a}=m\frac{d\vec{v}}{dt}$$

$$\vec{F}\cdot\vec{v}=\frac{d}{dt}\left(\tfrac{1}{2}m\,\vec{v}\cdot\vec{v}\right)=\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)$$

$$\vec{F}\cdot\vec{v}=\frac{d}{dt}T$$

$$\int(\vec{F}\cdot\vec{v})\,dt=\int dt=T+\text{const.}$$

$$\int(\vec{F}\cdot\vec{v})\,dt=\int(F_x\dot{x}+F_y\dot{y}+F_z\dot{z})\,dt$$

$$=\int(F_x\,dx+F_y\,dy+F_z\,dz)$$

$$=\int\left(-\frac{\partial V}{\partial x}dx-\frac{\partial V}{\partial y}dy-\frac{\partial V}{\partial z}dz\right)$$

$$=-\int dV=-V+\text{const.}$$

$$-V+\text{cosnt}=T+\text{const.}$$

$$T+V=\text{const.}$$

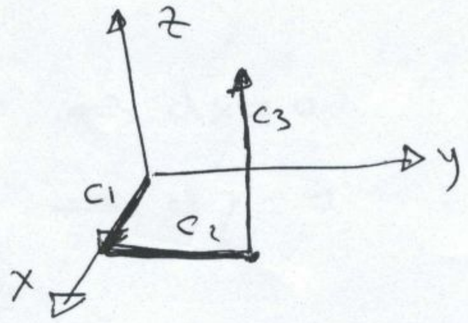
س/ لدينا القوة التي مركباتها تعطى بالعلاقة التالية.

$$F_x=ay(y^2-3z^2)$$

$$F_y=3ax(y^2-z^2)$$

$$F_z=-6axyz$$

5) برهن ان القوة $\vec{F}$ محافظة؟ ثم جد دالة الجهد باتباع المسار



الحل

$$V(x,y,z) = -\int F\cdot dr$$

$$= -\int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr - \int_{C_3} F\cdot dr$$

① $\int_{C_1} F\cdot dr$

$dy = dz = 0 \qquad \therefore y = z = 0$

$F_x = F_y = F_z = 0$

$dr = \hat{x}\, dx \Rightarrow -\int_{C_1} F\cdot dr = 0$

② $\int_{C} F\cdot dr$

$x = \text{const}, \quad dx = 0$

$z = 0, \quad dz = 0$

$dr = \hat{y}\, dy$

$F_x = ay^3$

$F_y = 3axy^2$

$F_z = 0 \Rightarrow \int_{C_2} F\cdot dr = \int_{C_2} 3axy^2\, dy$

$= axy^3 + \text{const}$

(3) $\int_{C_3} F \cdot dr$

$x = \text{const} \Rightarrow dx = 0$

$y = \text{const} \Rightarrow dy = 0$

$dr = \hat{z} dz$

$-\int_{C_3} F_z dz = -\int 6axyz\,dz = -3axyz^2 + \text{const}.$

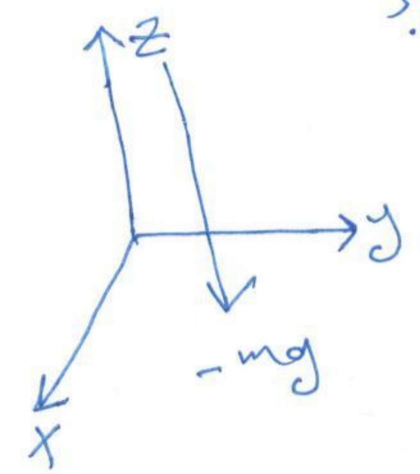
$$V = -axy^3 + 3axyz^2 + \text{const}.$$

★ تمارين

س ١/ جسم كتلته m يسقط سقوطاً حراً شاقولياً
بإتجاه المحور Z (أهمل مقاومة الهواء) جد مركبة القوة
المؤثرة على الجسم ومعادلة الطاقة الكامنة ومعادلة الطاقة الكلية ؟

E, V, T, F الحل

قوة الجاذبية قوة محافظة



$$\therefore F_x = -\frac{\partial V}{\partial X} = 0$$

$$\therefore F_y = -\frac{\partial V}{\partial X} = 0$$

$$\therefore F_z = -\frac{\partial V}{\partial Z} = -mg \Rightarrow V = \int mg\,dZ = mgZ$$

$$V = mgZ + \text{constant}$$

$$v^2 = \dot{Z}^2$$

$$T = \frac{1}{2}m\dot{Z}^2$$

$$E = V + T = mgZ + \frac{1}{2}m\dot{Z}^2 + \text{constant}$$

س٧/ جسم كتلته m يتحرك حركة توافقية بسيطة متناسقة في ثلاثة اتجاهات جد رسم القوة / الطاقة الكلية ، الطاقة الحركية ؟

الحل

$$f_x = -\frac{\partial V}{\partial x} , \quad f_x = -kx = -\frac{\partial V}{\partial x}$$

$$f_y = -\frac{\partial V}{\partial y} , \quad f_y = -ky = -\frac{\partial V}{\partial y}$$

$$f_z = -\frac{\partial V}{\partial z} , \quad f_z = -kz = -\frac{\partial V}{\partial z}$$

$$V(x,y,z) = \frac{1}{2} k(x^2+y^2+z^2) + \text{Constant}$$

$$V = \frac{1}{2} k r^2 + \text{Constant} , \quad T = \frac{1}{2} m v^2$$

$$E = T + V = \frac{1}{2} m v^2 + \frac{1}{2} k r^2 + \text{Constant}$$

س٨/ لدينا كتلة m تؤثر عليه دالة الجهد التالية :

$$V(x,y,z) = -\frac{k}{\sqrt{x^2+y^2+z^2}} = -\frac{k}{r}$$

جد مركبات القوة المؤثرة عليه f_x, f_y, f_z ثم برهن أن القوة المؤثرة عليه هي

$$\bar{F} = -\frac{k}{r^2}\hat{r}$$

Sol.

$$f_x = -\frac{\partial V}{\partial x} , \quad f_y = -\frac{\partial V}{\partial y} , \quad f_z = -\frac{\partial V}{\partial z}$$

$$\frac{\partial V}{\partial x} = \frac{1}{2} \frac{k(2x)}{(x^2+y^2+z^2)^{3/2}} = \frac{kx}{r^3}$$

$$f_x = -\frac{kx}{r^3} , \quad f_y = -\frac{ky}{r^3} , \quad f_z = -\frac{kz}{r^3}$$

$$\bar{F} = \hat{x} f_x + \hat{y} f_y + \hat{z} f_z = -\frac{k}{r^3}\left[\hat{x}x + \hat{y}y + \hat{z}z\right]$$

$$= -\frac{k}{r^3}\bar{r}$$

★(consevative force excerciese)

Excorsise

تدريب (مسألة)عن القوة المحافظة

1) a) find Curl $\bar{F}$ or $\nabla \times \bar{F}$

a) $\bar{F} = \hat{i}\, cyz + \hat{j}\, czx + \hat{k}\, cxy$

b) $\bar{F} = \frac{cy}{z}\hat{i} + \frac{cx}{z}\hat{j} + \frac{cxy}{z^2}\hat{k}$

c) $\bar{F} = \frac{k\, \bar{e}_r}{r^3}$

d) $\bar{F} = \hat{i}\, e^{a(x+y)} + \hat{j}\, e^{b(x+y)} + \hat{k}\, e^{cz}$

(2) find the force from the potential function

a) $V = k(x^2 + 2xy + y^2)$

b) $V = \frac{kxy}{z^2}$

c) $V = k e^{a(x^2+y^2+z^2)}$

d) $V = k(x+y+z)^n$

P21

امثلة

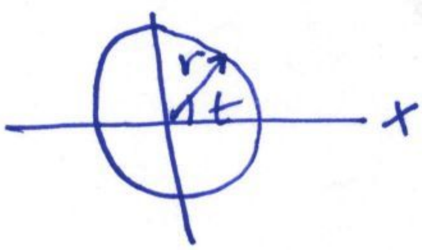
م 7

مثال: اوجد الشغل المبذول لتحريك جسم لمرة واحدة حول دائرة C في المستوي xy اذا كان مركز الدائرة يقع في نقطة الاصل وكان نصف قطرها 3 والقوة

$$\bar{F} = (2x - y + z)i + (x + y - z^2)j + (3x - 2y + 4z)k$$

$$\bar{r} = xi + yj$$

$$d\bar{r} = dx\,i + dy\,j$$



$$\int_C \bar{F} \cdot d\bar{r} = \int \left[(2x - y + z)i + (x + y - z^2)j + (3x - 2y + 4z)k \right] \cdot \left[dx\,i + dy\,j \right]$$

$z = 0$

$$= \int \left[(2x - y)\,dx + (x + y)\,dy \right]$$

$y = 3\sin t$, $x = 3\cos t$, $dx = -3\sin t\,dt$, $dy = 3\cos t$

$t \to 0 \to 2\pi$

$$\int_{t=0}^{2\pi} [6\cos t - 3\sin t][-3\sin t\,dt] + [3\cos t + 3\sin t][3\cos t]\,dt$$

$$\int_0^{2\pi} -18\cos t \sin t\,dt + \int_0^{2\pi} 9\sin^2 t\,dt + \int_0^{2\pi} 9\cos^2 t\,dt + \int_0^{2\pi} 9\sin t \cos t\,dt$$

$= W$

$$= (-18 + 9)\int_0^{2\pi} \cos t \sin t\,dt + 9(2\pi)$$

$$-9\,\frac{\sin^2 t}{2}\Big|_0^{2\pi} + 9(2\pi) = \boxed{18\pi}$$

P12

س١/ اذا كانت $\bar{F} = -\nabla V$ بين ان الشغل المبذول في ازاحة جسم من نقطة $P_1(x_1,y_1,z_1)$ الى نقطة $P_2(x_2,y_2,z_2)$ لا يعتمد على المسار الذي يصل بين النقطتين.

ج/

$$w = \int_{P_1}^{P_2} \bar{F}\cdot dr = -\int_{P_1}^{P_2} \nabla V \cdot dr$$

$$= -\int_{P_1}^{P_2} \left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k \right) (dx\, i + dy\, j + dz\, \bar{k})$$

$$= -\int_{P_1}^{P_2} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = -\int_{P_1}^{P_2} dV$$

$$= V(P_1) - V(P_2) = V(x_1,y_1,z_1) - V(x_2,y_2,z_2)$$

س٣/ اثبت ان

$$\bar{F} = (2xy + z^3) i + x^2 j + 3xz^2 k$$

A) قوة لا نسفاً "

B) اوجد الجهد

C) اوجد الشغل المبذول لنقل الجسم من النقطة $(1,-2,1)$ الى النقطة $(3,1,4)$

ج/ A)

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0$$

B)

$$\bar{F} = -\nabla V$$

$$-\nabla V = -\frac{\partial V}{\partial x} i - \frac{\partial V}{\partial y} j - \frac{\partial V}{\partial z} \bar{k} = (2xy + z^3) i + x^2 j + 3xz^2 k$$

$$\frac{\partial V}{\partial x} = 2xy + z^3, \quad \frac{\partial V}{\partial y} = x^2, \quad \frac{\partial V}{\partial z} = 3xz^2$$

P3

$$W=\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{(1,-2,1)}^{(3,1,4)} (2xy+z^3)\,dx + \int_{(1,-2,1)}^{(3,1,4)} x^2\,dy$$

$$+\int_{(1,-2,1)}^{(3,1,4)} 3xz^2\,dz$$

$$= \left[x^2y+z^3x\right] + \left[x^2y\right] + \left[xz^3\right] \Bigg|_{(1,-2,1)}^{(3,1,4)}$$

$$= \left[9+192\right]+\left[9\right]+\left[192\right]$$

$$-\left[-2+1\right]+\left[-2\right]+\left[1\right] =$$

$$= 402 - 0 = 402$$

س٢ | اثبت أنه اذا كانت F هي القوة المؤثرة على جسيم ما في v فان القدرة تعطى

$$\bar{P}=\bar{F}\cdot\bar{v}$$

12

$$dw=\bar{F}\cdot d\bar{r}$$

$$P=\frac{dw}{dt}=\bar{F}\cdot\bar{v}$$

اذا كانت

$$\bar{F}=60t\,i+(180t^2-10)\,j-120\,k$$

$$\bar{v}=(6t^2\pm1)\,i+(12t^3-2t)\,j-24t\,k$$

P4

مثال / يتحرك جسم كتلته 5kg تحت تأثير قوة ذات متجه الموضع

$$\bar{r} = (2t^3+t)\,i + (3t^4 - t^2 + 8)\,j - 12t^2\,k$$

أوجد A) السرعة B) الزخم C) التعجيل D) القوة

A) $\bar{v} = \frac{\partial \bar{r}}{\partial t}$ B) $p = m\bar{v}$ C) $\bar{a} = \frac{\partial^2 r}{\partial t^2}$

$$F = m\bar{a} = m\frac{\partial \bar{v}}{\partial t} = m\frac{\partial^2 \bar{r}}{\partial t^2}$$

مثال / جسم كتلته m يتحرك في المستوي xy

$$\bar{r} = a\cos wt\, i + b\sin wt\, j$$

A) بين ان الجسم يتحرك في قطع ناقص

B) ان القوة المؤثرة على الجسم تكون دائما متجهة نحو نقطة الاصل

ج/

$$r = xi + yj = a\cos wt\, i + b\sin wt\, j$$

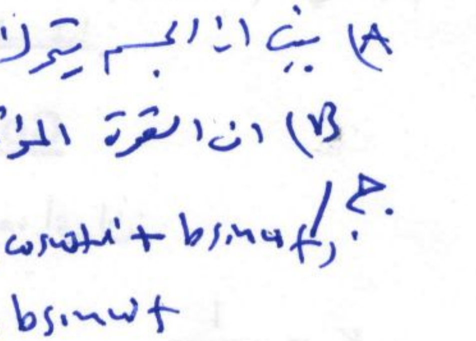
$$x = a\cos wt \quad , \quad y = b\sin wt$$

معادلة القطع الناقص

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\sin^2 wt + \cos^2 wt = 1$$

∴ تحقق



(B

$$\bar{F} = m\frac{dv}{dt} = m\frac{\partial^2 r}{\partial t^2}$$

$$= m[-w^2 a\cos wt\, i - w^2 b\sin wt\, j]$$

$$= m[-w^2 a\cos wt\, i - w^2 b\sin wt\, j]$$

$$= m[-w^2 a\cos wt \ldots mw^2 \ldots$$

هذا يعني ان القوة دائما باتجاه نقطة الاصل

PS

حل

جسم كتلته 2kg يتحرك تحت تأثير قوة

$$\bar{F} = 24t^2 i + (36t-16) j - 12t k$$

وأن الجسم عند اللحظة $t=0$

$r_0 = 3i - j + 4k$

وسرعته

$V_0 = 6i + 15j - 8k$

اوجد A) السرعة B) التعجيل عند اللحظة t

A)

$$m\frac{dw}{dt} = 2\frac{dw}{dt} = 24t^2 i + (36t-16) j - 12t k$$

$$\frac{dw}{dt} = 12t^2 i + (18t-8) j - 6t k$$

$$V = \int (12t^2 i + (18t-8) j - 6t k$$

$$V = \frac{12t^3}{3} i + (9t^2 - 8t) j - 3t^2 k + C_1$$

نطبق الشروط الحدودية

$t = 0$

$$\boxed{6i + 15j - 8k = C_1}$$

So

$$V = \left(\frac{12t^3}{3} + 6\right) i + (9t^2 - 8t + 15) j - (3t^2 + 8) k$$

اوجد

$\bar{r}$?

P26

مثال/

قوة ثابتة F تؤثر على جسم كتلته m فتغير السرعة

V_1 الى V_2 خلال زمن t !

$F = m(V_2 - V_1)/t$ أثبت أن

الحل.

من قانون نيوتن الثاني

$$F = m\frac{dv}{dt} \rightarrow F\,dt = m\,dv$$

تكامل الطرفين

$$\int_0^t F\,dt = m\int_{V_1}^{V_2} dv$$

وبما أن F ثابتة

$$Ft = m(V_2 - V_1)$$

$$F = \frac{m(V_2 - V_1)}{t}$$

س/ هل التكامل الشرطي تغير اذا كانت F متغيرة

ليست ثابتة ؟

س/ ما هي القوة الثابتة اذا كانت القوة مسلطة على

شاحنة قدرها $2000\,kg$ سرعتها $60\,km/hr$ الى حالة

السكون خلال 4 ثواني ؟

$$F = \frac{m(V_2 - V_1)}{t}, \quad t = 4$$

$V_2 = 0, \; V_1 = 60$

P7

س١/

جسم كتلته ثابتة m يتحرك في الفراغ تحت تأثير مجال القوة $\bar{F}$ اذا كانت السرعة v_1 , v_2 عند اللحظتين t_1 , t_2 على التوالي

اثبت ان الشغل $W = T_2 - T_1$

ج/

$$\int_{t_1}^{t_2} \bar{F}\cdot dr = \int_{t_1}^{t_2} \bar{F}\cdot\frac{dr}{dt}\cdot dt = \int_{t_1}^{t_2} \bar{F}\cdot\bar{v}\, dt$$

$$= \int_{t_1}^{t_2} m\frac{dv}{dt}\cdot\bar{v}\, dt = m\int_{t_1}^{t_2}\frac{dv}{dt}\cdot v\, dt$$

$$= m\int_{t_1}^{t_2} v\cdot dv = m\frac{v^2}{2}\Big|_{t_1}^{t_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= T_2 - T_1$$

س٢/ أوجد الشغل المبذول لتحريك جسم على المنحنى $\bar{r} = 3i + 2j - 5k$ بواسطة القوة $\bar{F} = 2i - j - k$

$$W = \bar{F}\cdot\bar{r} = (2)(3) - 2 + 5 = 6 - 2 + 5 = 9$$

س٣/ أوجد الشغل المبذول على جسم كتلته $m = 1000\,kg$ وتزداد سرعته من $v_1 = 105\times10^3\,cm/sec$ الى $v_2 = 3\times10^3\,cm/sec$

ج/

$$W = T_2 - T_1 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

P8

س١١ /

اثبت ان مجال القوة F المعرفة بالصيغة

$$F = (y^2z^3 - 6xz^2)i + 2xyz^3 j + (3xy^2z^2 - 6x^2z)k$$

هو مجال قوة محافظ.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$f_x = y^2z^3 - 6xz^2$$

$$f_y = 2xyz^3 \qquad f_z = 3xy^2z^2 - 6x^2z$$

س١٢ / اذا كانت $V = 3x^2z^2 - xy^2z^3$

اوجد الشغل المبذول في تحريك الجسم من نقطة

$A(-2,1,3)$ $B(1,-2,-1)$

$$W = \int_A^B F \cdot dr = \int_A^B -\nabla V \cdot dr$$

$$= -\int_A^B \left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k \right) \cdot (dx\,i + dy\,j + dz\,k)$$

$$= -\int_A^B dV = -V(x,y,z) \Big|_{(-2,1,3)}^{(1,-2,-1)}$$

$$= xy^2z^3 - 3x^2z^2 \Big|_{(-2,1,3)}^{(1,-2,-1)} = ?$$

P19

س١٢ / اثبت أن $\bar{f} = -m\omega^2 \bar{r}$ قوة محافظة ؟

$\bar{r} = x\hat{i} + y\hat{j}$

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -m\omega^2 x & -m\omega^2 y & 0 \end{vmatrix} = 0$$

س١٤ / اثبت أن

$$\int_{t_1}^{t_2} F\,dt = mv_2 - mv_1 = p_2 - p_1$$

الدفع = الزخم

ج /

$$\int_{t_1}^{t_2} F\,dt = \int_{t_1}^{t_2} \frac{d}{dt}(mv)\,dt = \int_{t_1}^{t_2} d\,mv = mv \Big|_{t_1}^{t_2}$$

$$= mv_2 - mv_1 = p_2 - p_1$$

س١٥ / تتحرك كتلة مقدارها 5 kg على خط مستقيم وتتغير السرعة من 540 km/hr الى 720 km/hr في زمن قدره دقيقتان . ما هو الدفع خلال هذا الزمن

$$\int_{t_1}^{t_2} F\,dt = m(v_2 - v_1) =$$

س١٦ / اثبت أن

$$\bar{r} \times \bar{F} = \frac{d}{dt}\{m(\bar{r} \times \bar{v})\}$$

عزم القوة حول نقطة الأصل

ج /

$$\bar{r} \times \bar{F} = \bar{r} \times \frac{d}{dt}(m\bar{v})$$

$$= m(\bar{r} \times \frac{\partial}{\partial t}\bar{v}) = m\frac{\partial}{\partial t}(\bar{r} \times \bar{v})$$

$$\bar{r} \times \bar{F} = \frac{\partial}{\partial t}\{m(\bar{r} \times \bar{v})\}$$

$$\frac{d}{dt}\{m(\bar{r} \times \bar{v})\} = m\bar{r} \times \frac{\partial \bar{v}}{\partial t}$$

$= 0$

م ١٠/٢

١٧ أوجد العزم $\bar{A} = \bar{r} \times \bar{F}$ والزخم الزاوي

$\Omega = \bar{r} \times \bar{P}$

$\bar{r} = (t^4 + 6t + 3)i + (8t^3 - 4t^2 + 15t - 1)j + (4 - t^3 - 8t)k$

$\bar{F} = 24t^2 i + (36t - 16)j - 12tk$

$$A = \bar{r} \times \bar{F} = \begin{vmatrix} i & j & k \\ t^4 + 6t + 3 & 8t^3 - 4t^2 + 15t - 1 & 4 - t^3 - 8t \\ 24t^2 i & 36t - 16 & -12t \end{vmatrix} =$$

الزخم الزاوي

$$\Omega = \bar{r} \times m\bar{v}$$

$$= m \begin{vmatrix} i & j & k \\ r_x & r_y & r_z \\ v_x & v_y & v_z \end{vmatrix} =$$

---

١٨ يتحرك جسم في مجال القوة $\bar{F} = r^2 \bar{r}$ حيث r ثابت فيه

برهن ان الجسم ان $\bar{A} = \bar{r} \times \bar{F}$ تكون ثابتة

ج/

$\bar{A} = \bar{r} \times \bar{F} = \bar{r} \times (r^2 \bar{r})$

$= r^2 \{\bar{r} \times \bar{r}\} = 0$ العزم المؤثر على الجسم

P:11

سؤال 19 / اثبت ان مجال القوة

$\bar{F} = x^2 y z i - x y z^2 k$ هو مجال غير محافظ

س / اوجد الشغل المبذول بواسطة القوة وكذلك الزخم

والدفع ... من النقطة $t=1$ الى النقطة $t=2$ وكذلك الزخم

$V = (4t^3+6)i + (9t^2-8t+15)j$

$+ (3t^2+8)k$ / ح

$t_1 = 1$ الحل

$V_1 = 10i + (16)j + 11\bar{k}$

$V_2 = 38i + (35)i + 20k$

$W = T_2 - T_1 = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2$

$V_2^2 = \bar{V}_2 \cdot \bar{V}_2 = 100 + 16*16 + 11*11$

$V_1^2 = \bar{V}_1 \cdot \bar{V}_1 =$

الزخم

$P = mV$

P_1 and P_2

$t=1$ $t=2$

$\int f \cdot dt = \int_1^2 [24t^2 i + \text{الدفع}] dt$

P:12

س١/ جسيم كتلته m يتحرك على المحور x تحت تأثير مجال قوة كامنة جهده V(x) اذا كان الجسم يشغل الموضع x_1 و x_2 عند اللحظتين t_1، t_2 اثبت ان

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

ج/

$$E = \frac{1}{2} m \left(\frac{\partial x}{\partial t}\right)^2 + V(x)$$

$$E - V(x) = \frac{1}{2} m \left(\frac{\partial x}{\partial t}\right)^2$$

$$\left(\frac{\partial x}{\partial t}\right)^2 = \frac{E - V(x)}{\frac{1}{2} m} \rightarrow \frac{\partial x}{\partial t} = \sqrt{\frac{2(E - V(x))}{m}}$$

$$\frac{\partial t}{\partial x} = \sqrt{\frac{m}{2(E - V(x))}}$$

$$\int dt = \int \frac{\sqrt{\frac{m}{2}}\, dx}{\sqrt{E - V(x)}} \rightarrow t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V}}$$

س٢/ احسب $t_2 - t_1$ اذا تحرك الجسم من الموضع $x = 0$ الى $x = 10$ وان الطاقة الكامنة $V(x) = \frac{1}{2} k x^2$

ج/

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_0^{10} \frac{dx}{\sqrt{E - \frac{1}{2} k x^2}}$$

استخدم التكامل

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int_0^{10} \frac{dx}{\sqrt{\frac{1}{2} k \left(\frac{2}{k} E - x^2\right)}} = \sqrt{\frac{2}{k}} \int_0^{10} \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \sqrt{\frac{2}{k}} \sin^{-1} \frac{x}{\sqrt{\frac{2E}{k}}} \Bigg|_0^{10}$$

2E

P 13

مسائل اضافية

① جسم كتلته m يتحرك في مسار فراغي

$$\bar{r} = (4t^2 - t^3)\, i - 5t\, j + (t^4 - 2)\, k$$

اوجد عند $t = 1$

A) الزخم

B) القوة

ج/

$$F = 4i + 24k$$

$$P = 10i - 10j + 8k$$

---

② دالة الزخم لجسم

$$p = 3e^{-t}\, i - 2\cos t\, j - 3\sin t\, k$$

اوجد F ؟

$p = mv$

$v = \frac{p}{m}$

$F = m\frac{\partial v}{\partial t}$

ج/

$$F = -3e^{-t}\, i + 2\sin t\, j - 3\cos t\, k$$

---

③ دالة الجسم يسير في فضاء

$$r = a\cos\omega t\, i + b\sin\omega t\, j$$

اثبت ان

$\bar{r} \times \bar{p} = mab\omega\, \bar{k}$

$\bar{r} \cdot \bar{p} = \frac{1}{2} m (b^2 - a^2) \sin 2\omega t$

$\bar{r} \times \bar{f} = 0$

$\bar{p} = m\frac{\partial r}{\partial t}$

$$\bar{r} \times \bar{p} = \begin{vmatrix} i & j & k \\ a\cos\omega t & b\sin\omega t & 0 \\ p_x & p_y & 0 \end{vmatrix}$$

---

④ قوة مقدارها 100N في الاتجاه الموجب للمحور x تؤثر على جسم كتلته 2kg لمدة 10 ثواني . ما هي السرعة التي يكتسبها الجسم بفرض انه يبدأ من السكون $v_i = 0$ ؟

$$F = \frac{m(v_2 - v_1)}{t}, \quad v_2 ?$$

P: 13

مثال٢: جسم كتلته 3kg يتحرك في المستوي xy تحت تأثير جهد $V = 12x(3y - 4x)$ في اللحظة t=0 يبدأ الجسم في الحركة من السكون الذي منه موضع $10i - 10j$

A) اوجد المعادلات التفاضلية B) حل المعادلات C) اوجد الموضع في أي لحظة D) اوجد السرعة في أي لحظة؟

ج / بما أن الجهد

$$V = 12x(3y - 4x)$$
$$= 36xy - 48x^2$$

$$F = -\nabla V = -\frac{\partial V}{\partial x} i - \frac{\partial V}{\partial y} j - \frac{\partial V}{\partial z} k$$
$$= -(36y - 96x) i - 36x j - 0$$

$$m \frac{\partial^2 r}{\partial t^2} = 3 \frac{\partial^2 r}{\partial t^2} = -(36y - 96x) i - 36x j$$

$$3 \frac{\partial^2 r}{\partial t^2} = -(36y - 96x) i - 36x j$$

وبما أن $r = xi + yj$

$$3 \left[\frac{\partial^2 x}{\partial t^2} i + \frac{\partial^2 y}{\partial t^2} j \right] = -(36y - 96x) i - 36x j$$

$$3 \frac{\partial^2 x}{\partial t^2} = -(36y - 96x) \Rightarrow \frac{\partial^2 x}{\partial t^2} = -(12y - 32x)$$

$$3 \frac{\partial^2 y}{\partial t^2} = -36x \Rightarrow \frac{\partial^2 y}{\partial t^2} = -12x$$

x ? y ?

بما أن

$$r = 10i - 10j$$

$$v = \frac{\partial r}{\partial t} = 0$$

P114

40kg

⑤ اوجد القوة الثابتة اللازمة لتغيير سرعة جسم كتلته
بحيث تتغير سرعتها من $4i - 5j + 3k$ m/sec إلى
$8i + 3j - 5k$ m/s خلال 10 ثانية

ح/

$$F = \frac{m(V_2 - V_1)}{t}$$

$$\bar{F} = \frac{40(4i - 2j - 8k)}{10\ sec} =$$

وما مقدار القوة

$$|F| ?$$

---

⑥ جسم كتلته 1kg يتحرك في مجال قوة معطاة بدلالة الزمن t على الصورة

$$\bar{F} = (6t - 8)i - 60t^3 j + (120t^3 + 36t^2)\hat{k}$$

الموضع والسرعة الابتدائية

$$r_0 = 2i - 3k$$

$$V_0 = 5i + 4j$$

A) موضع الجسم

B) سرعة الجسم

$t = 2$

ح/

$$m\frac{\partial v}{\partial t} = (6t - 8)i - 60t^3 j + (120t^3 + 36t^2)\hat{k}$$

$$\int dv = V = \int (6t - 8)i\,dt - \int 60t^3 j\,dt + (120t^3 + 36t^2)\hat{k}\,dt$$

$$\bar{V} = (3t^2 - 8t)i - 15t^4 j + [5t^4 + 12t^3]k + V_0$$

$$\bar{V} = (3t^2 - 8t + 5)i - (15t^4 - 4)j + [5t^4 + 12t^3]k$$

$$r ?$$

P1   8 م

مسائل إضافية

س١/ أثبت أن

$$F = (y^2\cos x + z^3)\,i + (2y\sin x - 4)\,j + (3xz^2 + 2)\,k$$

هي قوة محافظة (A) أوجد الجهد المصاحب لمجال القوة (B)

(C) أحسب الشغل المبذول في إزاحة الجسم من النقطة $(0,1,-1)$ إلى $(\frac{\pi}{2}, -1, 2)$

ج/

A. $V = y^2\sin x + xz^3 - 4y + 2z + c$

الشغل B) $15 + 4\pi$ $\quad F = -\nabla V, \quad w = \int_{(-1,1,0)}^{(\frac{\pi}{2},-1,2)} f\cdot dr$

س٢/ أثبت أن القوة $\bar{F} = r^5\,\bar{r}$ هي قوة محافظة

$\bar{r} = xi + yj + zk$

(B) أوجد الجهد المصاحب

ج/

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$V = -\frac{1}{4} r^4 + c$$

س٣/ أحسب الجهد المصاحب للقوة $\bar{F} = -k r^{-n}\,\bar{r}$

ج/

~~V =~~

$$-\nabla V = -k r^{-n}\,\bar{r}$$

$$\bar{r} = xi + yj + z\bar{k}$$

$$-\left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k\right) = -k r^{-n}(xi + yj + zk)$$

$$\frac{\partial V}{\partial x} = k r^{-n} x, \quad \frac{\partial V}{\partial y} = k r^{-n} y, \quad \frac{\partial V}{\partial z} = k r^{-n} z$$

$$\partial V = k r^{-n} x\, dx$$

$$V = \int k r^{-n} x\, dx \Rightarrow V_x = k\int \frac{x\,dx}{(x^2+y^2+z^2)^{\frac{n}{2}}}$$

$$V_x = \frac{k}{2}\,(x^2+y^2+z^2)^{-\frac{n}{2}+1}$$

$$V(x,y,z) = V_x + V_y + V_z$$

$V_y =$ ——— , V_z

P3

س٣/ ما ... الطاقة القدرة الناتجة في مجال القوة

احسب الشغل $p(t) = 3t^2 - 4t + 2$
المبذول في ازاحة الجسم من النقطة ... الى النقطة $t = 2$
$t = 4$

36 / ج.

$\frac{dw}{dt} = \partial p(t)$

$p(t) = \frac{\partial w}{\partial t}$

$dw = p(t)\,dt \Rightarrow w = \int_2^4 (3t^2 - 4t + 2)\,dt$

$w = t^3 - 2t^2 + 2t \Big|_2^4 = 64 - 32 + 8 - 8 + 8 - 4$

$= 36$

س٤/ قوة تزيح جسم ... m على المسار
$r = a\cos wt\, i + b\sin wt\, j$ ما هي القدرة؟

$P = F \cdot \bar{v}$ ج/

س/ ان الزخم الزاوي لجسم ... بدلالة t على الصورة

$\Omega = 6t^2 i - (2t + 1) j + (12t^3 - 8t^2) k$ ... $t = 1$
احسب العزم ... اللحظة $t = 1$

ج/

$\Omega = \bar{r} \times \bar{p} = \bar{r} \times m\bar{v}$

$A = \bar{r} \times \bar{F}$?

$A = \frac{\partial \Omega}{\partial t} = \bar{r} \times m \frac{\partial \bar{v}}{\partial t}$

$= 12t\, i - 2j + (36 - 16)$

$= 12i - 2j + 20k$

P3

س١ / إذا وضع الجسم تحت تأثير قوة $\bar{F} = 100te^{-2t}i$

A) أوجد التغير في الزخم من $t=1$ إلى $t=2$

B) الدفع

ج / $P = mv$, $f = m\frac{\partial v}{\partial t}$

$$\bar{F} = \frac{\partial p}{\partial t} = m\frac{\partial v}{\partial t}$$

$$100te^{-2t}i = \frac{\partial p}{\partial t} \quad , P = \int_{t_1}^{t_2} 100te^{-2t}i\,dt$$

$$\Delta\bar{P} = 100\int_1^2 te^{-2t}i\,dt$$

$$udv = uv - \int v\,du \quad , u = t \quad du = dt$$

$$dv = e^{-2t} \quad V = -2e^{-2t}$$

$$-2te^{-2t}\Big|_1^2 + 2\int_1^2 e^{-2t}\,dt = -2te^{-2t}\Big|_1^2 + e^{-2t}\Big|_1^2$$

$$= -4e^{-4} + 2e^{-2} + e^{-4} - e^{-2}$$

$$-3e^{-4} + e^{-2}$$

س٢ / جسم يتحرك مرة واحدة حول الدائرة

$$\bar{r} = a(\cos\phi i + \sin\phi j)$$

$$\bar{F} = (xi - yj)/(x^2 + y^2)$$

A) أوجد الشغل B) هل مجال القوة محافظ

س/ جسيم كتلته 2 كيلوغرام في مجال القوة:

$$f = 8xyi + (4x^2 - 8z)j - 8yk \quad Newton$$

إذا كانت سرعته 4 m/sec عند النقطة (-1,2,-1)

فكم تكون سرعته عند (1,-1,1) ج/

~~$\frac{\partial v}{\partial t} = 4xyi + (\frac{4}{2}x^2 - \frac{8}{2}z)j - 4yk$~~ $= v\frac{dv}{\partial x}$

~~$V =$~~

$$p = \bar{f} \cdot \bar{V} =$$

~~$\bar{p} =$~~ $|f| \cdot |\bar{V}| =$

$$\bar{f} = -16i + (4+8)j - 16\bar{k}$$

$$|\hat{f}| = \sqrt{(-16)^2 + (12)^2 + (-16)^2}$$

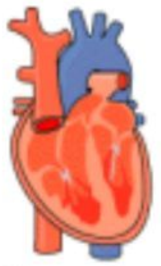
$$p = |\vec{f}| |\bar{v}| = 4\sqrt{\quad}$$

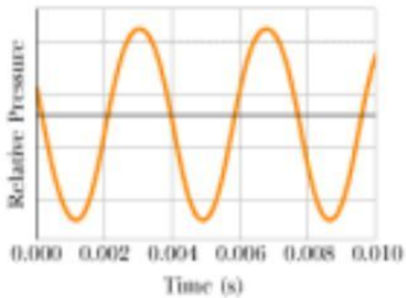
$$p = |\vec{f}|_2 \cdot \vec{V} \;?$$

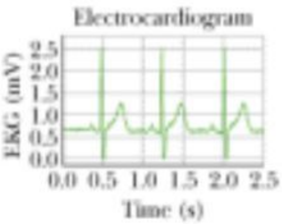
م 9

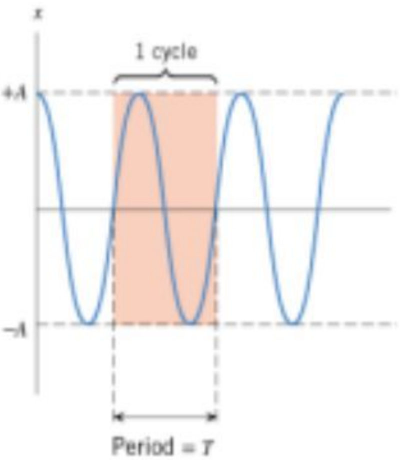
The periodic motion

Any measurable quantity that repeats itself at regular time intervals is defined as undergoing **periodic** motion.
If the periodic variation of a physical quantity over time has the shape of a sine (or cosine) function, we call it a **sinusoidal oscillation or simple harmonic motion.**



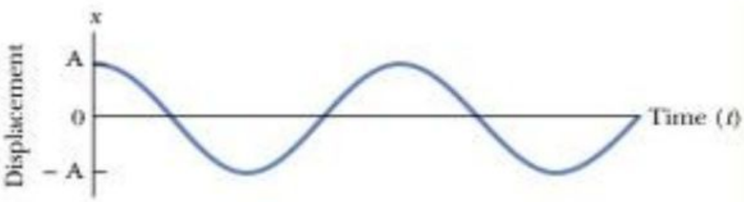


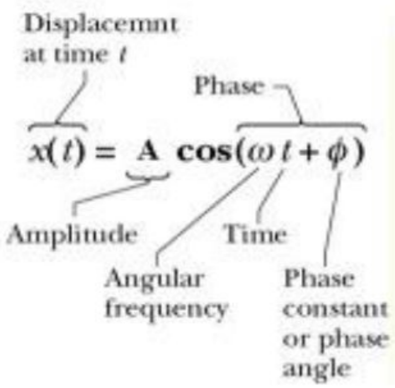




- The **period T** is the time required for one complete motional cycle.
- The **frequency f** of the motion is the number of cycles of the motion per second (unit is: 1 cycle/second=1 Hz).
- Frequency and period are related according to: $f = \frac{1}{T}$

Simple Harmonic Motion



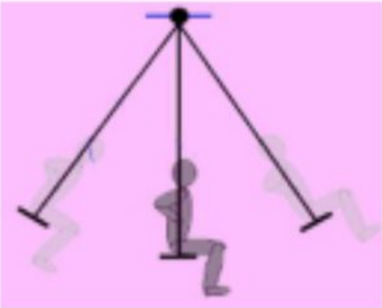


$$T = \frac{2\pi}{\omega}$$

$$T = \frac{1}{f}$$

Note: Angles are in radians.

Introduction to Oscillations



- In order to describe the complicated forms of periodic motion around us, usually we start by an analysis of the simplest form of oscillations,

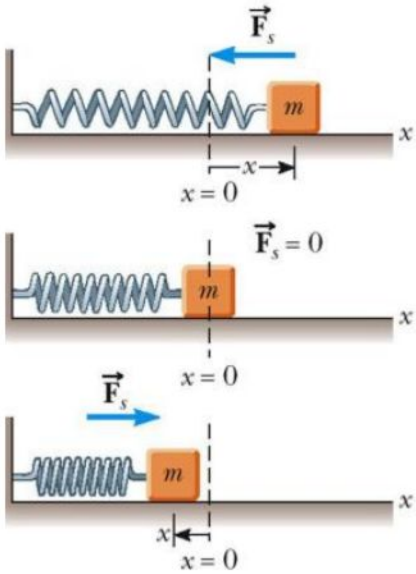
the simple harmonic motion.

- The main two characteristics of the simple harmonic motion are;

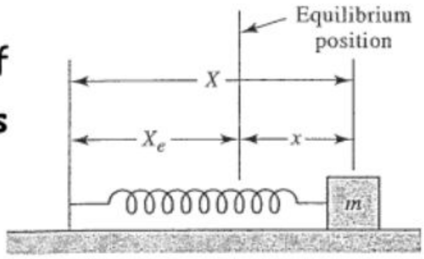
(1) It is described by a second-order, linear differential equation with constant coefficients. Thus, the Superposition principle holds, if two particular solutions are found, their sum is also a solution.

(2) It has Amplitude-independent periods .That is, the periodic time of the motion, is independent of the maximum displacement from equilibrium (the amplitude) .

3.2. Linear Restoring Force: Harmonic Motion



- Consider a mass m on a frictionless surface attached to a wall by means of a spring.
- Let X_e is the unstretched length of the spring. **This position represents the equilibrium position where the potential energy is a minimum.**



- If the mass is pushed or pulled away from this position, the spring will be either compressed or stretched, and then exert a force on the mass.
- This force will always attempt to restore it to its equilibrium position.
- To calculate the motion of the mass, we need an expression for this *restoring force* .

According to *Hooke's law* The spring's restoring force is given by ;

$$F(x) = -kx$$

where k is the spring constant. In fact, this law is valid only for small displacements from equilibrium, where the restoring force is **linear**.

Newton's second law of motion can now be written as

$$m\ddot{x} + kx = 0 \qquad (3.2.4a)$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad (3.2.4b)$$

looking for a solution which can show that **the motion is both periodic and bounded**. *Sine* and *cosine* functions both can exhibit that sort of behavior. Thus, a possible solution is;

$$x = A\sin(\omega_0 t + \phi_0) \qquad (3.2.5)$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad (3.2.6)$$

is the *angular frequency* of the system.

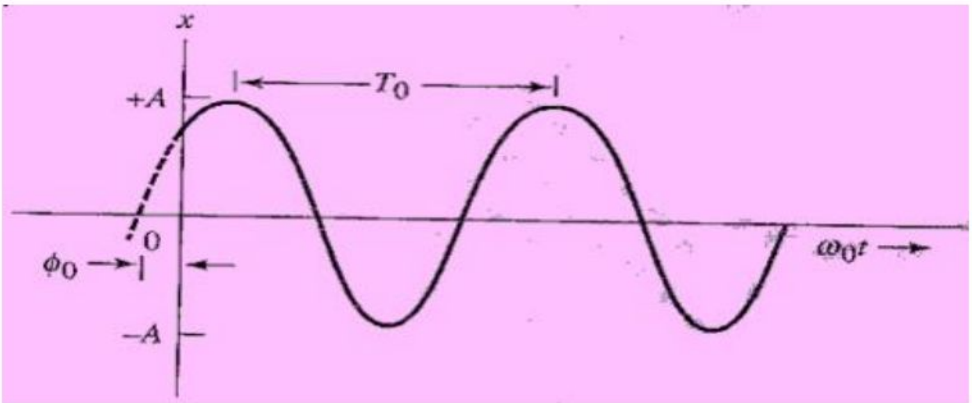


Figure 3.2.3 Displacement versus $\omega_0 t$ for the simple harmonic oscillator.

The motion exhibits the following features:

(1) The motion repeats itself after a time T_0 known as *the period* of the motion. Which is *the time required for a phase advance of 2π*, and is given by

$$\omega_0(t + T_0) + \phi_0 = \omega_0 t + \phi_0 + 2\pi$$

Or;

$$T_0 = \frac{2\pi}{\omega_0}$$

(2) The motion is bounded; that is, it is confined within the limits $-A \le x \le +A$. Where, A is called *the amplitude* of the motion and it is **independent** of ω_0.

(3) The phase angle ϕ_0 is the initial value of the sine function. It determines the value of the displacement x at time $t = 0$. I.e, $x(t=0) = A\sin(\phi_0)$

(4) The term *frequency*, f_0, refer to the reciprocal of the period of the oscillation or

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The unit of frequency (*cycles per second*, or s^{-1}) is called the hertz (Hz).

(5) Constants of the Motion A and ϕ_0, can be determined from the *initial conditions* as follows:

$$x(0) = A\sin(\phi_0) = x_0$$
$$\dot{x}(0) = \omega_0\, A\cos(\phi_0) = v_0$$
$$\therefore \quad \tan\phi_0 = \frac{\omega_0 x_0}{v_0}$$

$$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2}$$

Simple Harmonic Motion: The Projection of a Rotating Vector

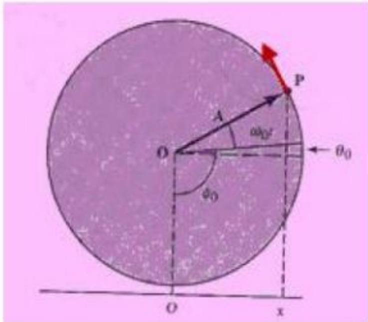
Imagine a vector **A** rotating at a constant angular velocity w_o . Let this vector denote the position of a point P moves in uniform circular motion.

The **projection of A** traces out simple harmonic motion.

Since $\dot{\theta} = \omega_0$ **and** $\theta = \omega_0 t + \theta_0$

θ_0 is the value of θ at t=0.

Thus, the projection of P onto the x-axis is



$$x = A\cos\theta = A\cos(\omega_0 t + \theta_0)$$

Or with the equivalence expression:

$$x = A\sin(\omega_0 t + \phi_0)$$

where $\phi_0 - \theta_0 = \pi/2$

$$\cos(\omega_0 t + \theta_0) = \cos\left(\omega_0 t + \phi_0 - \frac{\pi}{2}\right) = \sin(\omega_0 t + \phi_0)$$

Or we could represent the general solution for harmonic motion:

$$\begin{aligned} x &= A\sin\phi_0 \cos\omega_0 t + A\cos\phi_0 \sin\omega_0 t \\ &= C\cos\omega_0 t + D\sin\omega_0 t \end{aligned}$$

Note that: $\tan\phi_0 = \dfrac{C}{D}$, $A^2 = C^2 + D^2$

Effect of a Constant External Force on a Harmonic Oscillator

Suppose the same spring shown in Figure 3.2.1 is held in a vertical position, supporting the same mass m (Fig. 3.2.5). The total force acting is now given by adding the weight mg to the restoring force,

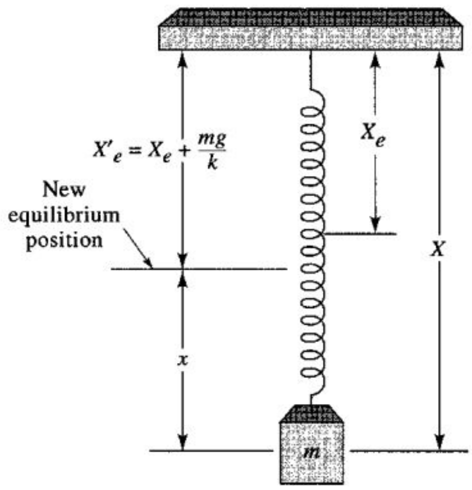
$$F = -k(X - X_e) + mg \tag{3.2.20}$$

This equation could be written $F = -kx + mg$

where, $x = X - X_e$

let $\acute{X}_e$ be the displacement from the new equilibrium position obtained by setting $F = 0$

$0 = -k(X'_e - X_e) + mg$, which gives $X'_e = X_e + mg/k$.



We now define the displacement as

$$x = X - X'_e = X - X_e - \frac{mg}{k}$$

$$F = -kx \quad (3.2.22)$$

so the differential equation of motion is again

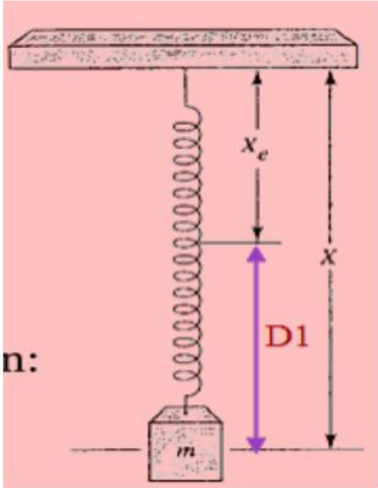
$$m\ddot{x} + kx = 0 \quad (3.2.23)$$

and our solution in terms of our newly defined x is identical to that of the horizontal case.

Example (3.2.1): Effect of a Constant External Force

When a light spring supports a block of mass m vertically, the spring is found to stretch by an amount D1 over its unstreched length. If the block is furthermore pulled downward a distance D2 then released at time **t = 0**, find:

(a) the resulting motion.
(b) the velocity of the block at the equilibrium position.
(c) the acceleration of the block at the top of its oscillatory motion.



n:

Solution:

Assuming the positive direction is down, at the equilibrium position we have

$$F_x = 0 = -kD_1 + mg$$

This gives us the value of the spring constant k:

$$k = \frac{mg}{D_1}$$

From this we can find the angular frequency of oscillation:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$

We will express the motion in the form

Then,
$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\dot{x} = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$$

Applying the initial conditions , we find

$x_0 = D_2 = A$ & $v_0 = 0 = B\omega_0 \Longrightarrow B = 0$

The motion is, therefore, given by

(a) $$x = D_2 \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

Note that the mass m does not appear in the final expression.
The velocity is then

(b) $$\dot{x} = -D_2\sqrt{\frac{g}{D_1}}\sin\left(\sqrt{\frac{g}{D_1}}t\right) \xrightarrow{\text{(center)}} \dot{x} = -D_2\sqrt{\frac{g}{D_1}}$$

and the acceleration

(c) $$\ddot{x} = -D_2\frac{g}{D_1}\cos\left(\sqrt{\frac{g}{D_1}}t\right) \xrightarrow{\text{(top)}} \ddot{x} = D_2\frac{g}{D_1}$$

In the case $D_1 = D_2$ the downward acceleration at the top of the swing is just g.
This means that the block, at that particular instant, is in free fall; that is, the spring is exerting zero force on the block.

Example (3.2.2):

The simple pendulum consists of a small mass m swinging at the end of a light string of length l. The motion is along a circular arc defined by the angle θ, as shown in the Fig.

The restoring force is ; $F_s = -mg \sin\theta$.
Therefore, the differential equation of motion is;

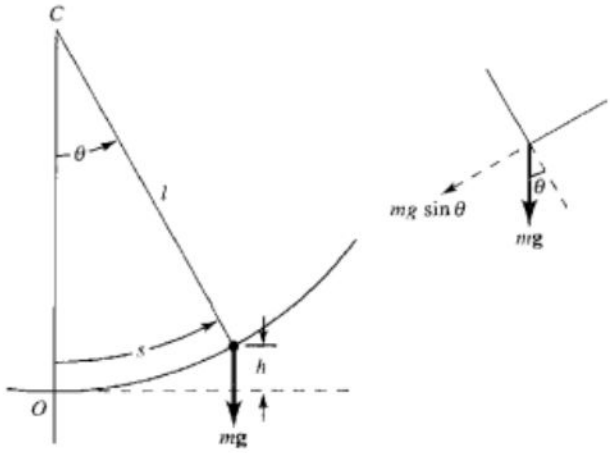
$$m\ddot{s} = -mg\sin\theta$$

Since $s = l\theta$ and, for small θ, $\sin\theta = \theta$, we can write the differential equation of motion as follows:

or

$$\ddot{s}+\frac{g}{l}s=0$$

$$\ddot{\theta}+\frac{g}{l}\theta=0$$



Although the motion is along a curved path, the differential equation is mathematically identical to that of the linear harmonic oscillator;

$$\ddot{x}+\frac{k}{m}x=0$$

Thus, for the angles that the approximation sin θ = θ is valid, we can conclude that the motion is simple harmonic with angular frequency

$$\theta = \theta_0 \cos(\omega_0 t + \phi_0)$$

$$\omega_0=\sqrt{\frac{g}{l}}$$

and period

$$T_0=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{l}{g}}$$

3.3 Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate **the work** done by an external force F_{ext} in moving the particle from the equilibrium position ($x = 0$) to some position x. We have, $F_{ext} = -F_x = kx$, so

$$W = \int_0^x F_{ext} dx = \int_0^x kx\, dx = \frac{k}{2} x^2$$

This work is stored in the spring as potential energy: $V(x)$, where

$$V(x) = \tfrac{1}{2} kx^2$$

Thus, $F_x = -dV/dx = -kx$, as required by the definition of V. The total energy, when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely,

$$E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 \tag{3.3.3}$$

The motion of the particle can be found by starting with the energy equation (3.3.3). Solving for the velocity gives

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{kx^2}{m}\right)^{1/2} \tag{3.3.4}$$

which can be integrated to give t as a function of x as follows:

$$t = \int \frac{dx}{\pm[(2E/m) - (k/m)x^2]^{1/2}} = \mp(m/k)^{1/2} \cos^{-1}(x/A) + C \tag{3.3.5}$$

in which C is a constant of integration and A is the amplitude given by

$$A = \left(\frac{2E}{k}\right)^{1/2} \tag{3.3.6}$$

We also see from the energy equation that the maximum value of the speed, which we call v_{max}, occurs at $x = 0$. Accordingly, we can write

$$E = \tfrac{1}{2} mv_{max}^2 = \tfrac{1}{2} kA^2 \tag{3.3.7}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where $x = 0$ and $\dot{x} = \pm v_{max}$, and it is all potential energy at the extrema, where $\dot{x} = 0$ and $x = \pm A$.

EXAMPLE 3.3.1

The Energy Function of the Simple Pendulum

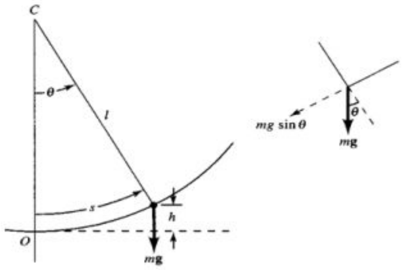
The potential energy of the simple pendulum (Fig. 3.2.6) is given by the expression

$$V = mgh$$

where h is the vertical distance from the reference level (which we choose to be the level of the equilibrium position). For a displacement through an angle θ (Fig. 3.2.6), we see that $h = l - l\cos\theta$, so

$$V(\theta) = mgl(1 - \cos\theta)$$

Now the series expansion for the cosine is $\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \cdots$, so for small θ we have approximately $\cos\theta = 1 - \theta^2/2$. This gives



$$V(\theta) = \tfrac{1}{2} mgl\,\theta^2$$

or, equivalently, because $s = l\theta$,

$$V(s) = \tfrac{1}{2}\frac{mg}{l}s^2$$

Thus, to a first approximation, the potential energy function is quadratic in the displacement variable. In terms of s, the total energy is given by

$$E = \tfrac{1}{2} m\dot{s}^2 + \tfrac{1}{2}\frac{mg}{l}s^2$$

in accordance with the general statement concerning the energy of the harmonic oscillator discussed above.

EXAMPLE 3.3.2

Calculate the average kinetic, potential, and total energies of the harmonic oscillator. (Here we use the symbol K for kinetic energy and T_0 for the period of the motion.)

Solution:

$$\langle K\rangle = \frac{1}{T_0}\int_0^{T_0} K(t)\,dt = \frac{1}{T_0}\int_0^{T_0} \tfrac{1}{2} m\dot{x}^2\,dt$$

but

$$x = A\sin(\omega_0 t + \phi_0)$$
$$\dot{x} = \omega_0 A\cos(\omega_0 t + \phi_0)$$

Setting $\phi_0 = 0$ and letting $u = \omega_0 t = (2\pi/T_0)\cdot t$, we obtain

$$\langle K\rangle = \frac{1}{T_0}\left[\tfrac{1}{2} m\omega_0^2 A^2 \int_0^{T_0} \cos^2(\omega_0 t)\,dt\right]$$
$$= \frac{1}{2\pi}\left[\tfrac{1}{2} m\omega_0^2 A^2 \int_0^{2\pi} \cos^2 u\,du\right]$$

We can make use of the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}(\sin^2 u+\cos^2 u)\,du=\frac{1}{2\pi}\int_0^{2\pi}du=1$$

to obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 u\,du=\tfrac{1}{2}$$

because the areas under the $\cos^2$ and $\sin^2$ terms throughout one cycle are identical. Thus,

$$\langle K\rangle=\tfrac{1}{4}m\omega_0^2A^2$$

The calculation of the average potential energy proceeds along similar lines.

$$V=\tfrac{1}{2}kx^2=\tfrac{1}{2}kA^2\ \sin^2\omega_0 t$$

$$\begin{aligned}\langle V\rangle&=\tfrac{1}{2}kA^2\frac{1}{T_0}\int_0^{T_0}\ \sin^2\omega_0 t\,dt\\&=\tfrac{1}{2}kA^2\frac{1}{2\pi}\int_0^{2\pi}\ \sin^2 u\,du\\&=\tfrac{1}{4}kA^2\end{aligned}$$

Now, because $k/m=\omega_0^2$ or $k=m\omega_0^2$, we obtain

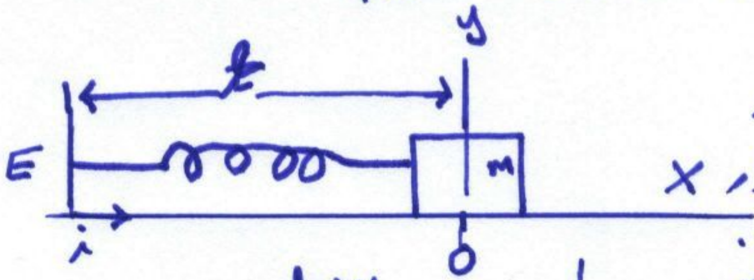
$$\langle V\rangle=\tfrac{1}{4}kA^2=\tfrac{1}{4}m\omega_0^2A^2=\langle K\rangle$$

$$\langle E\rangle=\langle K\rangle+\langle V\rangle=\tfrac{1}{2}m\omega_0^2A^2=\tfrac{1}{2}kA^2=E$$

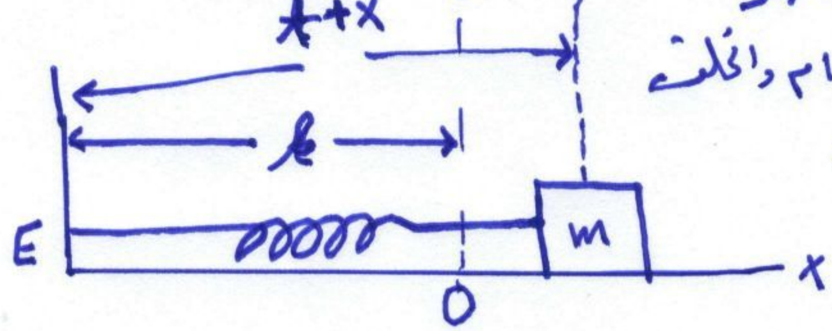
The average kinetic energies and potential energies are equal; therefore, the average energy of the oscillator is equal to its total instantaneous energy.

المتذبذب التوافقي / البندول البسيط

Harmonic Oscillation



الطول الطبيعي هو l
اذ سحبت الكتلة على طول محور x
ثم تركت فانها سوف تهتز
او تتذبذب الى الامام والخلف
حول موضع الاتزان O



ولايجاد معادلة الحركة ان طول السبرنك بعد السحب هو $l+x$
حيث توجد قوة تحاول ارجاع الكتلة الى موضع التوازن حسب قانون هوك Hooke law
تسمى هذه القوة بـ قوة الاسترداد او القوة المعيدة.
وتتناسب مع الاستطالة وتعطى بالعلاقة التالية

$$\vec{F}_R = -kx\hat{i}$$

حيث ان R يشير الى Return force (قوة الارجاع) وباستخدام قانون نيوتن الثاني

$$\vec{F} = m\ddot{x} = -kx\hat{i}$$

$$\text{or} \quad m\ddot{x} + kx\hat{i} = 0 \longrightarrow (1)$$

هذه تسمى معادلة المتذبذب التوافقي البسيط
وهذه الحركة تسمى الحركة التوافقية البسيطة.

السعة (A) Amplitude والزمن الدوري

periodic Time (T)

الآن نبدأ بحل المعادلة التفاضلية بمعرفة الشروط الحدودية

(ابتدائية)

$x = A$ when $\frac{\partial x}{\partial t} = 0$

$t = 0$

$\dot{x} = A$

$$\ddot{x} + \frac{K}{m} x = 0$$

$$\frac{\partial^2 x}{\partial t^2} + \frac{K}{m} x = 0 \quad \text{—} (1), \quad \omega = \frac{K}{m}$$

حل المعادلة

$$D^2 x + \omega^2 x = 0$$

$$D^2 = \mp \omega^2 \Rightarrow D = i\omega$$

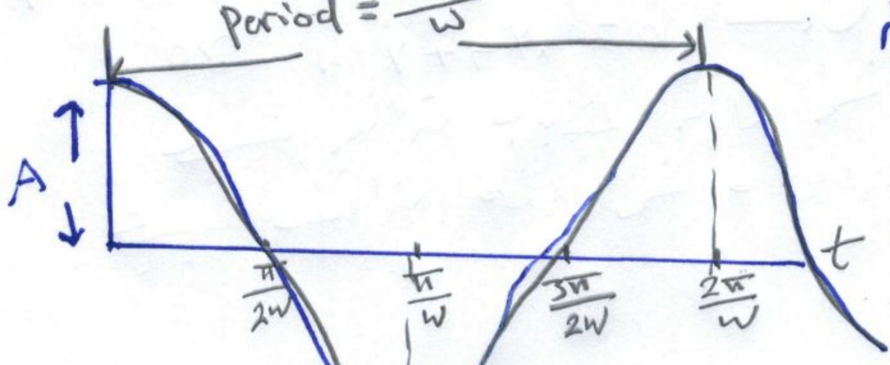
$$x = A e^{Dt} \Rightarrow x = A e^{i\omega t}$$

$$x = A \cos\omega t + iB \sin\omega t \quad , \quad \cos 0 = 1$$

at $t = 0$ $\quad \sin 0 = 0$

$$x = A \quad , \text{ so } B = 0$$

$$x = A\cos\omega t \quad , \omega = \sqrt{\frac{K}{m}} \quad \text{—} (2)$$



A : سعة الحركة

T : زمن ذبذبة كاملة

وإن الزمن الدوري

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}} \quad \text{—} (3)$$

وإن عدد الذبذبات أو الدورات الكاملة في وحدة الزمن يرمز له بالتردد

$$f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \cdot \sqrt{\frac{K}{m}} \quad \text{—} (4)$$

P3

ان المعادلة

$$x = A\cos(\omega t - \phi)$$

حيث ان A : السعة

$$\phi = \tan^{-1}\frac{B}{A}$$

ϕ : زاوية الطور ، وان قيمتها $0 \le \phi \le \pi$

وعندما $\phi = 0$

$$x = A\cos(\omega t) \quad \text{— (5)}$$

* طاقة المتذبذب التوافقي البسيط

Energy of Harmonic oscillator

T : الطاقة الحركية ، V طاقة الجهد

$$E = T + V \quad , \quad T = \frac{1}{2}mv^2 \; , \; V = \frac{1}{2}kx^2$$

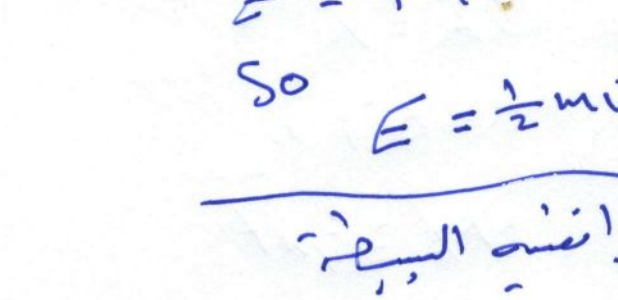
So

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

أسئلة ومسائل الحركة التوافقية البسيطة

① جسم كتلته 2 كيلوغرام يتحرك على المحور x ثم جذب نحو نقطة الأصل بقوة مقدارها 8x اذا كان الجسم في البداية ساكن عند x = 20 ، اوجد A) المعادلة التفاضلية والشروط الابتدائية لهذه الحركة B) موضع الجسم عند اي لحظة C) سرعة الجسم عند اي لحظة D) السعة والزمن الدوري.

ج/



A/

اعتبر $\bar{r} = x\hat{i}$ هو متجه الموضع

وان

$$\bar{a} = \frac{\partial^2 \bar{r}}{\partial t^2} = \frac{\partial^2 x}{\partial t^2}\hat{i}$$

وأن القوة المؤثرة على الجسم P $\bar{f} = -8x\hat{i}$

استخدام قانون نيوتن الثاني

$$\frac{\partial^2 x}{\partial t^2}\hat{i} = -8x\hat{i} \Rightarrow \boxed{\frac{\partial^2 x}{\partial t^2} + 8x = 0}$$

المعادلة التفاضلية المطلوبة

وتحقق الشروط الابتدائية

$$x = 20 \,,\; \frac{\partial x}{\partial t} = 0 \,,\; t = 0$$

(B

$$D^2 x + 8x = 0 \Rightarrow D^2 = -8$$

$$D = \mp i\,2\sqrt{2}$$

الحل

$$x = e^{\mp D t}$$

$$x = A\cos 2\sqrt{2}\,t + B\sin 2\sqrt{2}\,t$$

$$\frac{\partial x}{\partial t} = -2\sqrt{2}\,A\sin 2\sqrt{2}\,t + 2\sqrt{2}\,B\cos 2\sqrt{2}\,t$$

$$\frac{\partial x}{\partial t} = 0 \,,\; t = 0$$

$$0 = 0 + 2\sqrt{2}\,B \qquad \text{so } B = 0$$

$$\therefore x = A\cos 2\sqrt{2}\,t \qquad ,$$

$$v = \frac{\partial x}{\partial t} = -2\sqrt{2}\,A\sin 2\sqrt{2}\,t$$

وبأن $A = 20$

$$x = 20\cos 2\sqrt{2}\,t \quad , \quad \frac{\partial x}{\partial t} = -40\sqrt{2}\sin 2\sqrt{2}\,t$$

$$\omega = 2\pi f = 2\sqrt{2} = \sqrt{\frac{k}{m}}$$

$$f = \frac{\sqrt{2}}{\pi}$$

$$k^2 = m^2\omega = 4 * 2\sqrt{2} = 8\sqrt{2}$$

$$k = 2\sqrt{2}\,\sqrt{2} = 4$$

$\bar{f} = -8x\hat{i}$ وأن القوة المؤثرة على الجسم P

باستخدام قانون نيوتن الثاني

$$\frac{\partial^2 x}{\partial t^2}\hat{i} = -8x\hat{i} \Rightarrow \boxed{\frac{\partial^2 x}{\partial t^2} + 8x = 0}$$

المعادلة التفاضلية المطلوبة

وبتطبيق الشروط الابتدائية

$$x = 20 \;,\; \frac{\partial x}{\partial t} = 0 \;,\; t = 0$$

(B

$$D^2 x + 8x = 0 \Rightarrow D^2 = -8$$

$$D = \mp i 2\sqrt{2}$$

الحل

$$x = e^{\mp Dt}$$

$$x = A\cos 2\sqrt{2}\,t + B\sin 2\sqrt{2}\,t$$

$$\frac{\partial x}{\partial t} = -2\sqrt{2}A\sin 2\sqrt{2}\,t + 2\sqrt{2}B\cos 2\sqrt{2}\,t$$

$$\frac{\partial x}{\partial t} = 0 \;,\; t = 0$$

$$0 = 0 + 2\sqrt{2}B \quad \text{so } B = 0$$

$$\therefore x = A\cos 2\sqrt{2}\,t \quad ,$$

$$v = \frac{\partial x}{\partial t} = -2\sqrt{2}A\sin 2\sqrt{2}\,t$$

وبأن $A = 20$

$$x = 20\cos 2\sqrt{2}\,t \quad , \quad \frac{\partial x}{\partial t} = -40\sqrt{2}\sin 2\sqrt{2}\,t$$

$$\omega = 2\pi f = 2\sqrt{2} = \sqrt{\frac{k}{m}}$$

$$f = \frac{\sqrt{2}}{\pi}$$

$$k^2 = m^2\omega = 4 * 2\sqrt{2} = 8\sqrt{2}$$

$$k = 2\sqrt{2}\,\sqrt{2} = 4$$

P5

② جسم كتلته 20kg يتحرك على المحور X حركة توافقية بسيطة
في البداية t=0 كان الجسم على بعد 4m من نقطة الأصل وكانت سرعته 15m/sec وتعجيله $100\,m/sec^2$ متجهان نحو x=0 اوجد
A) الموضع X في أي لحظة B) سعة الذبذبة A وزمنها الدوري T والتردد f
C) القوة على الجسم عندما تكون $t = \frac{\pi}{10}$ sec

ج/

A) ان الشروط الابتدائية هي:

t=0 عند $x = 4$, $\frac{\partial x}{\partial t} = -15$, $\frac{\partial^2 x}{\partial t^2} = -100$

بما ان الحل العام للمعادلة هو

$$x = A\cos wt + B\sin wt \quad \text{---}(1$$

وان

$$\frac{\partial x}{\partial t} = -Aw\sin wt + Bw\cos wt \quad \text{---}(2$$

$$\frac{\partial^2 x}{\partial t^2} = -Aw^2\cos wt - Bw^2\sin wt \quad \text{---}(3$$

عند t=0 فان المعادلات الثلاث

$$4 = A + 0 \quad \text{---}(1$$

ينتج

$$A = 4m$$

$$-15 = -20w\sin wt + Bw\cos wt \quad \text{---}(2$$

عند t=0

$$-15 = Bw \rightarrow B = -\frac{15}{w}$$

$$-100 = -20w^2 + \frac{15}{w}w^2(0)$$

$$w^2 = \frac{100}{20} = 5 \rightarrow w^2 = \frac{k}{m}$$

$$w = \sqrt{5} = 2\pi f$$

P6 $\omega = \sqrt{5} = 2\pi f \rightarrow$

$f = \frac{\sqrt{5}}{2\pi}$, $T = \frac{1}{f} = \frac{2\pi}{\sqrt{5}}$

$A = 4$

ثانياً

$X = 20\cos\sqrt{5}\,t + \frac{15}{w}\sin wt$

$\frac{\partial X}{\partial t} = -20 * \sqrt{5}\,\sin wt - \frac{15}{w} * w\cos wt$

$\frac{\partial^2 X}{\partial t^2} = -20 * 5\cos\sqrt{5}\,t + \frac{15}{w} * w^2 \sin wt$

c) عند $t = \frac{\pi}{10}$ فان العجلة

$\bar{a} = \frac{\partial^2 X}{\partial t^2} = -100\cos\sqrt{5} * \frac{\pi}{10} + 15 * \sqrt{5}\sin\frac{\pi}{10}$

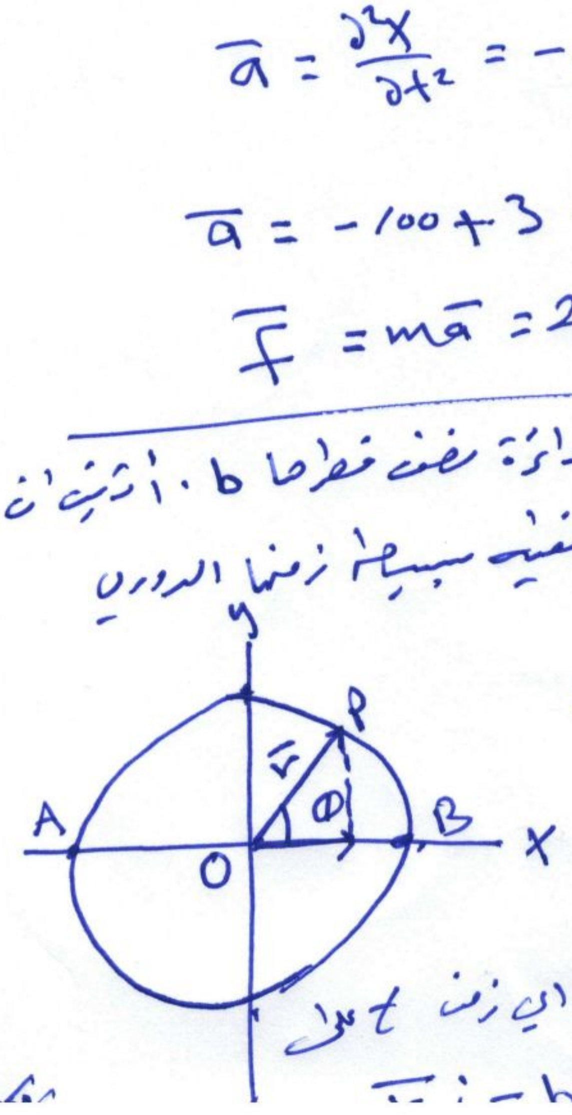
$\bar{a} = -100 + 3 = -97\ m/sec^2$

$\bar{F} = m\bar{a} = 20 * (-97) = 1940\ N$

---

(٣) جسم يتحرك بسرعة زاوية منتظمة w على دائرة نصف قطرها b . أثبت أن مسقط الجسم على القطر يتذبذب بحركة توافقية بسيطة زمنها الدوري $\frac{2\pi}{w}$ حول المركز.

ح/ تقع الدائرة في المستوى xy



$BOP = \Phi = wt$

$\bar{r} = b\cos wt\,i + b\sin wt\,j$ —①

المسقط Q على المحور x يكون عند اي زمن t على المسافة منه O

$\bar{x} = b\cos wt \rightarrow 2$

Problem

1) A guitar string vibrates harmonically with a frequency of 512 Hz (one octave above middle C on the musical scale). If the amplitude of oscillation of the center point of the string is 0.002 m (2 mm), what are the maximum speed and the maximum acceleration at that point?

يهتز وتر الكيتار بشكل متناغم بتردد ٥١٢ هرتز. إذا كانت سعة تذبذب نقطة مركز الخيط ٠,٠٠٢ م (٢ مم) ، فما أقصى سرعة وأقصى تسارع عند هذه النقطة

*Answer *

X(t)=Asin(*wt *), w=360 f

$$x = 0.002\sin\left[2\pi\left(512\,s^{-1}\right)t\right]\,[m]$$

$$\dot{x}_{\max} = (0.002)(2\pi)(512)\left[\frac{m}{s}\right] = 6.43\left[\frac{m}{s}\right]$$

$$\ddot{x}_{\max} = (0.002)(2\pi)^2(512)^2\left[\frac{m}{s^2}\right] = 2.07\times10^4\left[\frac{m}{s^2}\right]$$

**Q2 ** A piston executes simple harmonic motion with an amplitude of 0.1 m. If it passes through the center of its motion with a speed of 0.5 mIs, what is the period of oscillation?

Answer

يقوم المكبس بتنفيذ حركة توافقية بسيطة بسعة ٠,١ متر. إذا مرت عبر مركز حركتها بسرعة ٠,٥ ميكرومتر ، فما هي فترة التذبذب؟

X(t)=Asin(*wt *), w=360 f

$$x = 0.1\sin\omega_\circ t\ \ [m] \qquad \dot{x} = 0.1\,\omega_\circ\cos\omega_\circ t\left[\frac{m}{s}\right]$$

When $t = 0,\ x = 0$ and $\dot{x} = 0.5\left[\frac{m}{s}\right] = 0.1\,\omega_\circ$

$$\omega_\circ = 5\,s^{-1} \qquad T = \frac{2\pi}{\omega_\circ} = 1.26\,s$$

Q3\ A particle undergoes simple harmonic motion with a frequency of 10 Hz. Find the displacement x at any time t for the following initial condition:t=0 x=0.25m V=0.lm/s

يخضع الجسيم لحركة توافقية بسيطة بتردد ١٠ هرتز. أوجد الإزاحة في أي وقت

X(t) = A coswt+ Bsinwt

**Answer **

$$x(t) = x_{\circ} \cos \omega_{\circ} t + \frac{\dot{x}_{\circ}}{\omega_{\circ}} \sin \omega_{\circ} t \text{ and } \omega_{\circ} = 2\pi f$$

$$x = 0.25 \cos(20\pi t) + 0.00159 \sin(20\pi t)[m]$$

Q4 Verify the relations among the four quantities C, D, ϕ_0

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$x = A\cos(\omega_{\circ} t - \phi) = A\cos\phi \cos\omega_{\circ} t + A\sin\phi \sin\omega_{\circ} t$$

$$x = \text{A}\cos\omega_{\circ} t + \text{B}\sin\omega_{\circ} t ,\ \text{A} = A\cos\phi ,\ \text{B} = A\sin\phi$$

$$\boldsymbol{Tan\Theta = \frac{B}{A}}$$

Q5

A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1 and a velocity $\dot{x}_2$ when the displacement is x_2. Find the angular frequency and the amplitude of the motion in terms of the given quantities.

**Answer **

جسم يتذبذب حركة توافقية بسيطة بسرعة بسرعتين عند موقعين

اوجد السرعة الزاوية والسعة ؟

$$\frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} kx_1^2 = \frac{1}{2} m\dot{x}_2^2 + \frac{1}{2} kx_2^2$$

$$k(x_1^2 - x_2^2) = m(\dot{x}_2^2 - \dot{x}_1^2)$$

$$\omega_{\circ} = \sqrt{\frac{k}{m}} = \left(\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}}$$

$$\frac{1}{2} kA^2 = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} kx_1^2$$

$$A^2 = \frac{m}{k}\dot{x}_1^2 + x_1^2 = \frac{x_1^2\dot{x}_1^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} + x_1^2$$

$$A = \left(\frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2}\right)^{\frac{1}{2}}$$

Q6\

On the surface of the moon, the acceleration of gravity is about one-sixth that on the Earth. What is the half-period of a simple pendulum of length 1 m on the moon?

Answer\

على سطح القمر ان الجاذبية هي سدس الجاذبية على سطح الارض ما قيمة نصف دورة لبندول طوله متر واحد على القمر

$$\frac{1}{2}T_{\circ} = \pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{1}{\frac{9.8}{6}}}\ s \approx 2.5\,s$$

3.1 Introduction

Everywhere around us we see systems engaged in a ***periodic motion***: the small *oscillations* of a pendulum clock, a child playing on a *swing*, the rise and fall of the *tides*, the *swaying* of a tree in the wind and the *vibrations* of the strings on a violin. The essential feature that all these phenomena have in common is ***periodicity***, a pattern of movement or displacement that repeats itself over and over again.

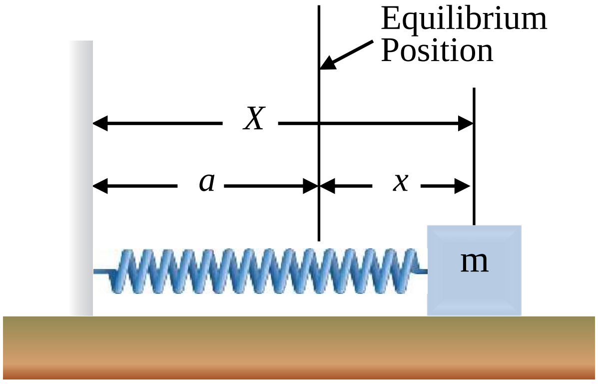
في كل مكان حولنا نرى أنظمة تعمل بحركة دورية: التذبذبات الصغيرة لساعة البندول و تأرجح الطفل وهو يلعب على أرجوحة وصعود وانحدار المد والجزر وتمايل الاشجار في الريح واهتزازات أوتار الكمان. الميزة الأساسية التي تشترك فيها كل هذه الظواهر هي الدورية، والتي هي نمط من الحركة أو الازاحة الذي يعيد نفسه مرارًا وتكرارًا.

3.2 Linear Restoring Force, Harmonic Motion

One of the most important cases of rectilinear motion is that produced by a ***linear restoring force***. This force whose magnitude is **proportional** to the **displacement** of the particle from the **equilibrium position** and whose **direction** is always **opposite** to that of the displacement. Such a force is exerted by a spring obeying Hooke's law.

واحدة من أهم حالات الحركة على خط مستقيم تلك التي تحدثها قوة معيدة خطية. هذه القوة التي يتناسب مقدارها مع إزاحة الجسم من موضع التوازن اتجاهها يكون دائماً عكس اتجاه الازاحة. قوة كهذه يسببها وتر أو نابض يخضع لقانون هوك.

$x = X - a$ ... ... (1)



Eq. (1) represents the displacement of the spring from its equilibrium length.

X is total length

a is upstretched (zero) length of the spring

Elastic force exerted by a spring obeying ***Hooke's law***

$F \alpha - x$

$F = -kx$ ……(2) ***Restoring Force* (*Hooke's law*)**

where k is called ***Stiffness* (*Spring Constant*)** معامل المرونة

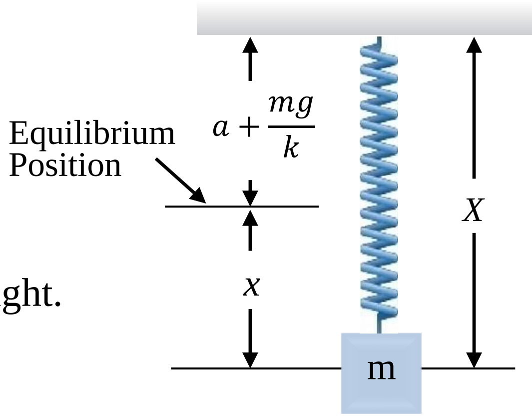
Sub. Eq.(1) in Eq.(2)

$\therefore F = -k(X - a) \dots \dots (3)$

Let the same spring be held vertically. The total force acting on the particle is:

$F = -kx$

$F = -k(X - a) + mg \dots \dots (4)$



where the *positive* direction is downward:

$x = X - \left(a + \frac{mg}{k}\right)$

$\frac{mg}{k}$ is the change in displacement due to body weight.

$\therefore x = X - a - \frac{mg}{k}$

$X - a = x + \frac{mg}{k} \dots \dots (5)$

Sub. Eq. (5) in Eq. (4)

$\therefore F = -k\left(x + \frac{mg}{k}\right) + mg$

$F = -kx - \frac{kmg}{k} + mg$

This give again:

$\therefore F = -kx$

$-kx = m\ddot{x}$

$\therefore m\ddot{x} + kx = 0 \dots \dots (6)$

Eq. (6) is **second order** differential Eq. with constant coefficients of the harmonic oscillator or linear oscillator. As equation $y'' + py' + qy = 0$, so, to solve such equation we shall employ the ***trail method*** in which the function (Ae^{qt}) is the trail solution, where (q) is a constant to be determined.

معادلة (6) هي معادلة تفاضلية من الدرجة الثانية ذات معاملات ثابتة لمذبذب توافقي أو مذبذب خطي. ويتم استخدام طريقة التجربة لحل المعادلة والذي فيه نجرب الحل (Ae^{qt}) حيث ان (q) ثابت.

$x = Ae^{qt}$ ... ... (7)

$\frac{dx}{dt} = \dot{x} = Aqe^{qt}$ ... ... (8)

$\frac{d^2x}{dt^2} = \ddot{x} = A\,q^2\,e^{qt}$ ... ... (9)

Sub. Eq. (7) and (9) in Eq. (6)

$m\,q^2\,A\,e^{qt} + kA\,q^{qt} = 0 \quad] \div A\,e^{qt}$

$mq^2 + k = 0$

$mq^2 = -k$

$q^2 = -\frac{k}{m}$

$\therefore q = \mp\sqrt{\frac{-k}{m}} = \mp i\sqrt{\frac{k}{m}}$

where $i = \sqrt{-1}$, $\sqrt{\frac{k}{m}} = w_0$

$q = \mp i w_0$ ... ... (10)

Sub. Eq. (10) in Eq. (7)

$\therefore x = Ae^{\mp i\,w_0 t}$ ... ... (11)

For a linear differential eqns., solution are additive, so, the general solution is:

$x = A_+\,e^{iw_0 t} + A_-\,e^{-iw_0 t}$ ... ... (12)

using ***Euler's formula***

$e^{iu} = \cos u + i \sin u$

So we can rewrite Eq. (12) in the form:

$x = A_+(\cos w_0 t + i \sin w_0 t) + A_-(\cos w_0 t - i \sin w_0 t)$

$= A_+ \cos w_0 t + i\,A_+ \sin w_0 t + A_- \cos w_0 t - i\,A_- \sin w_0 t$

$= (A_+ + A_-) \cos w_0 t + (iA_+ - iA_-) sin w_0 t$

$x = a \sin w_0 t + b \cos w_0 t$ ... ... (13)

where $a = i\,A_+ - i\,A_-$ and $b = A_+ + A_-$

The real solution of Eq. (13) is:

$x = b \cos w_0 t$

or $x = A \cos(w_0 t + \theta_o)$ ... ... (14) ***Sinusoidal Oscillation of Displacement x*** where: w_o is ***angular frequency*** التردد الزاوي

A is ***amplitude*** (the maximum value of x) السعة (اعظم قيمة للازاحة)

Equation (14) represent cosine function for two angle

So, $A\cos(w_0 t + \theta_o) = A(\cos w_0 t \cos\theta_o + \sin w_0 t \sin\theta_o)$

Let $a = A\cos\theta_o$, $b = A\sin\theta_o$

$$a^2 + b^2 = (A\cos\theta_o)^2 + (A\sin\theta_o)^2 = A^2(\cos^2\theta_o + \sin^2\theta_o)$$

$$A = (a^2 + b^2)^{1/2}$$

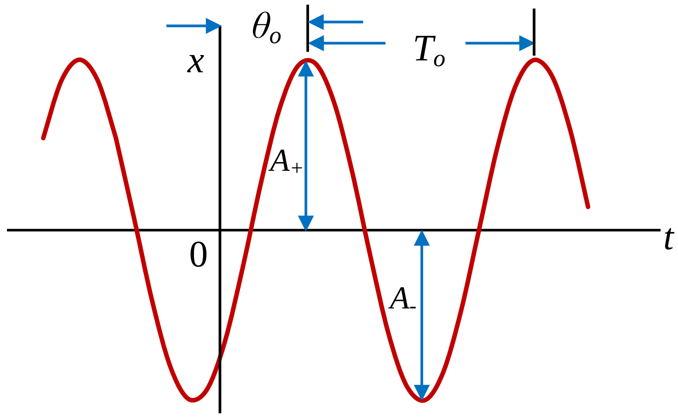
$$\frac{b}{a} = \frac{A\sin\theta_o}{A\cos\theta_o} = \tan\theta_o$$

$$\theta_0 = \tan^{-1}\left(\frac{b}{a}\right) \ldots\ldots(15)$$ ***Initial Phase***

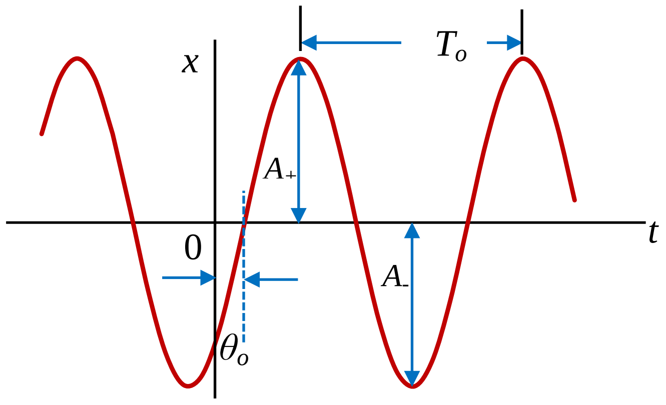
T_o is **time period** of the oscillation (time required for one complete cycle); that is, the period is the time for which the product (wt) increase by just (2π)

$$T_o = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}} \ldots\ldots(16)$$ زمن الذبذبة الزمن اللازم لدورة كاملة

$$w_0 = 2\pi f_0 \ldots\ldots(17)$$



$x = A\cos(w_o t + \theta_o)$



$x = A\sin(w_o t + \theta_o)$

$f_0 \equiv$ Linear frequency of oscillation (is the number of cycles in unit time) التردد الخطي للمتذبذب والذي يمثل عدد الدورات لوحدة الزمن

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ldots\ldots(18)$$

Example: محاضرة 12

A light spring is found to stretch an amount b when it supports a block of mass m. If the block is pulled downward a distance l from its equilibrium position and released at time $= 0$, find the resulting motion as a function of t.

Solution:

In the static equilibrium

$$F = -kb = -mg$$

$$\therefore k = \frac{mg}{b}$$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mb}} = \sqrt{\frac{g}{b}}$$

Now find the constants for the equation of motion

$$x = A\cos(w_0 t + \theta_0)$$

$$at \;\; t = 0 \,, \quad x = l \qquad \text{and } \dot{x} = 0$$

$$\dot{x} = -A\, w_0 \sin(w_0 t + \theta_0)$$

$$\dot{x} = -A\, w_0 \sin(w_0(0) + \theta_0) = 0$$

$$A\, w_0 \neq 0 \;\;,\;\; A = l \;\; is\ spring\ length, w_0\ is\ angular\ frequncy$$

$$A = l = X_{max}$$

$$\therefore \sin(\theta_0) = 0 \;\;\Rightarrow\;\; \theta_0 = 0$$

$$x = A\cos(w_0 t)$$

$$\therefore x = l\cos\left(\sqrt{\frac{g}{b}}\,t\right)$$

3.3 Energy Consideration in Harmonic Motion

Consider a particle moving under a linear restoring force$= -kx$. Let us calculate the work done by an external force f_a in moving the particle from the equilibrium position $x = 0$ to some position x.

نفترض جسيم يتحرك تحت تأثيرقوة معيدة خطية $-kx$ نحسب الشغل المنجز بواسطة قوة خارجية f_a لنقل الجسم من موضع التوازن $x = 0$ إلى موضع ما x

$$F = -kx \dots\dots (1)$$

$$F_a = -F = kx$$

$$\therefore W = \int F_a \, dx = \int_0^x kx \, dx = \frac{1}{2} kx^2$$

The work W is stored in the spring as potential energy الشغل يخزن في النابض كطاقة كامنة

$$\therefore V(x) = W = \frac{1}{2} kx^2 = E_p \dots\dots (2)$$

Total spring energy

$$E = E_k + E_p$$

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \quad \dots\dots (3) \qquad * \frac{2}{m}$$

$$\frac{2E}{m} = \dot{x}^2 + \frac{k}{m} x^2$$

$$\dot{x}^2 = \frac{2E}{m} - \frac{k}{m} x^2$$

$$\dot{x} = \left(\frac{2E}{m} - \frac{k}{m} x^2\right)^{1/2} \dots\dots (4)$$

This can be integrated to give (t) as function of x

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow dt = \frac{dx}{\dot{x}}$$

$$t = \int dt = \int \frac{dx}{\left(\frac{2E}{m} - \frac{k}{m} x^2\right)^{1/2}} \quad \dots\dots (5)$$

➤ ***Derive the Equation of Time***

- ***When $x = A\cos\theta$***

Rewrite Eq.(5)

$$t = \int \frac{dx}{\left(\frac{k}{m}\right)\left[\frac{2E}{k} - x^2\right]^{1/2}}$$

$$= \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{A^2 - x^2}} \quad \dots\dots (6)$$

where $A = \sqrt{\frac{2E}{k}} \equiv$ amplitude

$$x = A\cos\theta \dots\dots (7)$$

$$\therefore x^2 = A^2 \cos^2 \theta$$

$$\therefore A^2 - x^2 = A^2 - A^2 \cos^2 \theta$$

$$= A^2(1 - \cos^2 \theta)$$

$$A^2 - x^2 = A^2 \sin^2 \theta$$

$$\sqrt{A^2 - x^2} = A \sin \theta \; \dots \dots (8)$$

From Eq.(7)

$$dx = -A \sin \theta \; d\theta \; \dots \dots (9)$$

Sub. Eqns.(8) and (9) in Eq. (6)

$$t = -\sqrt{\frac{m}{k}} \int \frac{A \sin \theta}{A \sin \theta} \; d\theta$$

$$t = -\sqrt{\frac{m}{k}} \int d\theta$$

$$t = -\sqrt{\frac{m}{k}} \theta + c \; \dots \dots (10)$$

$$\because \; x = A \cos \theta$$

$$\therefore \cos \theta = \frac{x}{A} \Rightarrow \theta = \cos^{-1}\left(\frac{x}{A}\right) \dots \dots (11)$$

Sub. Eq.(11) in Eq. (10)

$$t = -\sqrt{\frac{m}{k}} \cos^{-1}\left(\frac{x}{A}\right) + c \; \dots \dots (12)$$

- $\boldsymbol{When\ x = A \sin \theta}$

$$\therefore \sin \theta = \frac{x}{A} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{A}\right) \dots \dots (a)$$

$$dx = A \cos \theta \; d\theta \; \dots \dots (b)$$

$$A^2 - x^2 = A^2 - A^2 \sin^2 \theta = A^2(1 - \sin^2 \theta)$$

$$A^2 - x^2 = A^2 \cos^2 \theta$$

$$\sqrt{A^2 - x^2} = A \cos \theta \; \dots \dots (c)$$

Sub. Eqns. (b) and (c) in Eq. (6)

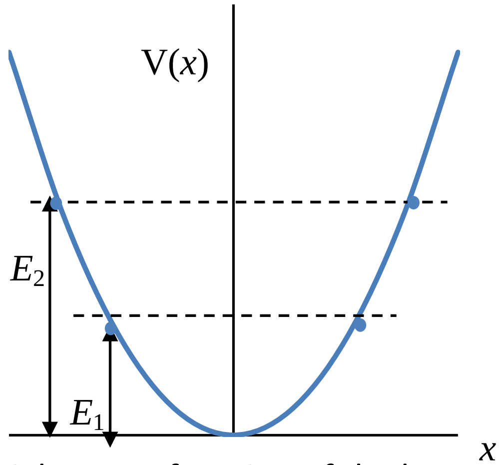
$$t = \sqrt{\frac{m}{k}} \int \frac{A\cos\theta}{A\cos\theta}\, d\theta$$

$$t = \sqrt{\frac{m}{k}}\theta + c \,\dots\dots (d)$$

Sub. Eq. (a) in Eq. (d)

$$t = \sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{A}\right) + c \,\dots\dots (13)$$

The value of x must lie between $\pm A\left(\pm\sqrt{\frac{2E}{k}}\right)$ in order for $\dot{x}$ to be real



Potential energy function of the harmonic oscillator.

1) at the upper point

$$x = X_{max}\,, v = 0$$

$$\therefore E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = 0 + \frac{1}{2}k$$

$$X_{max} = A$$

$$\therefore E = \frac{1}{2}kA^{\,2} \rightarrow A = \sqrt{\frac{2E}{k}}$$

2) at the lower point

$$x = 0\,, v = v_{max}$$

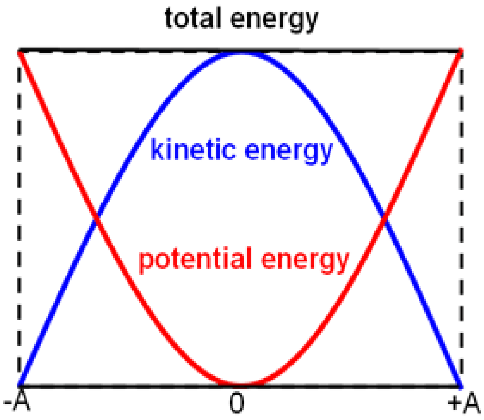
$$\therefore E = \frac{1}{2}\, mv_{max}^2$$

$$\frac{1}{2}m\, v_{max}^2 = \frac{1}{2}kA^2$$

$$v_{max}^2 = \frac{k\, A^2}{m}$$

$$v_{max} = \sqrt{\frac{k}{m}}A = Aw_0$$ ***Maximum Velocity for Harmonic Oscillator***

$$A = \frac{v_{max}}{w_0}$$



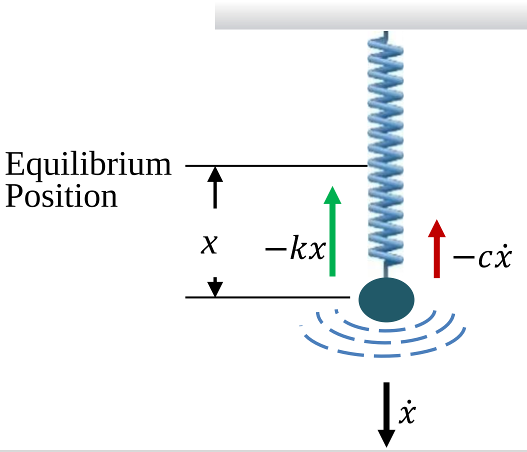
As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of ***kinetic energy at the center***, where $x = 0$ and $\dot{x} = \pm v_{max}$ and it is all ***potential energy at extrema***, where $x = \pm A$ and $\dot{x} = 0$.

عندما يتأرجح الجسيم، تتغير الطاقة الكامنة والحركية باستمرار. تكون الطاقة الكلية ثابتة حيث تكون بشكل طاقة حركية في المركز و طاقة كامنة عند النهايات.

3.4 Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is *idealized* in that we didn't take into account *frictional forces*. These are always present in any mechanical system. Consider an object is supported by a spring of stiffness k and there was a ***viscous retarding force*** varying linearly with the speed such as ***air resistance***.

التحليل السابق لمتذبذب توافقي مثالي لأنه لم ياخذ بنظر الاعتبار قوة الاحتكاك والتي تكون دائما موجودة في أي نظام ميكانيكي. لنفترض أن جسمًا ما معلقاً بنابض معامل مرونته k وكانت هناك قوة معيقة لزجة متغيرة خطياً مع السرعة مثل مقاومة الهواء.



$F = -kx$ …… (1) $\quad$ (***restoring force***)

$F = -c\,\dot{x}$ …… (2) $\quad$ (***retarding force***)

Equation of motion then:

$$\therefore -kx - c\dot{x} = m\ddot{x} \quad \ldots\ldots (3)$$

$m\ddot{x} + c\dot{x} + kx = 0$ … (4) ***Differential Eq. of Motion for Damped Harmonic Oscillator***

Use ***trail method*** to solve Eq. (4)

$$x = A\,e^{qt}$$

$$\therefore\ m\frac{d^2}{dt^2}(A\,e^{qt}) + c\frac{d}{dt}(A\,e^{qt}) + k\,A\,e^{qt} = 0$$

$$m\,q^2\,A\,e^{qt} + cq\,A\,e^{qt} + k\,A\,e^{qt} = 0]\ \div Ae^{qt}$$

$mq^2 + cq + k = 0$ …… (5) ***Auxiliary Equation***

$$ax^2 + bx + c = 0$$

$$x = \frac{-b\mp\sqrt{b^2 - 4ac_1}}{2a}$$

$$q = \frac{-c\mp\sqrt{c^2 - 4mk}}{2m} \ldots\ldots (6)$$

- $c^2 > 4mk$ **(*Over Damping*)**

  Here q will be real and negative and the motion will be nonoscillatory $q_1 \neq q_2$ and (x) decaying to zero exponentially with time.

المقدار $\boldsymbol{c^2 - 4mk}$ يحدد نوع التذبذب

$$q = -\begin{cases}\gamma_1\\ \gamma_2\end{cases} \quad \Rightarrow x = \begin{cases}A_1e^{-\gamma_1 t}\\ A_2e^{-\gamma_2 t}\end{cases}$$

حالة $\boldsymbol{c^2 > 4mk}$ تمثل حالة فوق التضاؤل وعندها q تمتلك قيمتين حقيقيتين سالبتين مختلفتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة x أسياً الى الصفرمع الزمن

  The general solution for displacement is:

$$x = A_1\,e^{-\gamma_1 t} + A_2\,e^{-\gamma_2 t} \ldots\ldots (7)$$

- $c^2 = 4mk$ **(*Critical Damping*)**

  Here q will be real also, and negative and the motion will be nonoscillatory. (x) decaying to zero exponentially with time but in shorter time.

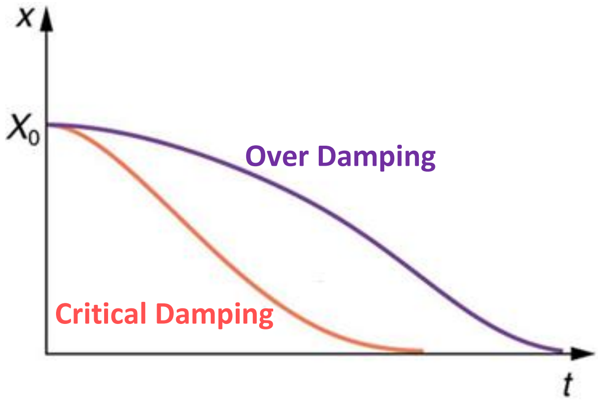
$$q_1 = q_2 = -\frac{c}{2m}$$

$$q = -\gamma \quad \Rightarrow x = \begin{cases}A_1e^{-\gamma t}\\ A_2te^{-\gamma t}\end{cases}$$

The general solution for displacement is:

$$x = A_1\, e^{-\gamma t} + A_2 t\, e^{-\gamma t} \;\ldots\ldots (8)$$

$$x = e^{-\gamma t}(A_1 + tA_2) \;\ldots\ldots (9)$$



حالة $c^2 = 4mk$ تمثل حالة التضاؤل الحرج وعندها q تمتلك قيمتين حقيقيتين سالبتين متساويتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة x أسياً الى الصفرمع الزمن

➢ $c^2 < 4mk$ **(*Under Damping*)**

Here q will be complex; the real part of its value gives an oscillatory motion.

$$q = \frac{-c \mp \sqrt{c^2 - 4mk}}{2m}$$

$$q = \frac{-c \mp \sqrt{\left(c^2 \cdot \frac{4m^2}{4m^2} - 4mk \cdot \frac{4m^2}{4m^2}\right)}}{2m}$$

$$q = \frac{-c \mp \sqrt{4m^2\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m} = \frac{-c \mp 2m\sqrt{\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m}$$

$$\gamma = \frac{c}{2m} \rightarrow \gamma^2 = \frac{c^2}{4m^2}, \quad w_0 = \sqrt{\frac{k}{m}}$$

$$q = \frac{-c \mp 2m\sqrt{\gamma^2 - w_0^2}}{2m}$$

$$q = -\frac{c}{2m} \mp \sqrt{\gamma^2 - w_0^2}$$

$$q_{1,2} = -\frac{c}{2m} + i\sqrt{w_0^2 - \gamma^2} = -\gamma \mp i\, w_1$$ ***Complex Conjugates Roots***

where $w_1 = \sqrt{w_0^2 - \gamma^2}$ is ***Natural Frequency***

$$q_1 = -\gamma + i\, w_1$$

$$q_2 = -\gamma - i\, w_1$$

The Displacement then:

$$\therefore x = A_+\, e^{(-\gamma + i\, w_1)t} + A_-\, e^{(-\gamma - i\, w_1)t} \;\ldots\ldots (10)$$

$$\therefore x = e^{-\gamma t}\left(A_+\, e^{i\, w_1 t} + A_-\, e^{-i w_1 t}\right)$$

$$e^{i\, u} = \cos u\, t + i \sin u$$ ***Euler's Formula***

$$x = e^{-\gamma t}\,[(i\,A_+ - i\,A_-)\sin w\,t + (A_+ + A_-)\cos w_1 t\,]$$

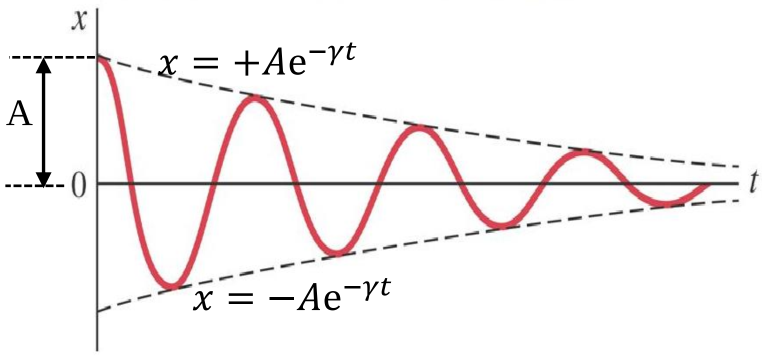
$$x = e^{-\gamma t}\,(\,a\sin w_1 t + b\cos w_1 t\,)$$

where $a = i\,(A_+ - A_-),\quad b = A_+ + A_-$

or $x = A\,\mathrm{e}^{-\gamma t}\cos(w_1 t + \theta_0) \ldots\ldots (11)$

where $\theta_0 = \tan^{-1}\frac{b}{a}$

$$A = (a^2 + b^2)^{1/2}$$



Equation (11) shows that the ***two curves*** are given by $x = +A\mathrm{e}^{-\gamma t}$ and $x = +A\mathrm{e}^{-\gamma t}$ form an ***envelope*** of the curve of motion because the cosine factor takes on values between +1 and -1, including +1 and -1, at which points the curve of motion touches the envelope. Accordingly, the points of contact are separated by a time interval of ***one-half period***.

في حالة $c^2 < 4mk$ والتي تمثل حالة دون التضاؤل عندها نحصل على قيمتين غير حقيقيتين (خيالية) لـq والحركة هنا تكون تذبذبية والسعة تتضائل اسياً مع الزمن.

تظهر المعادلة (11) وجود منحنيتين هما $x = +A\mathrm{e}^{-\gamma t}$ و $x = +A\mathrm{e}^{-\gamma t}$ يشكلان غلافاً لمنحنى الحركة لأن عامل الجيب تمام يأخذ القيم بين +1 و -1 ، بضمنها +1 و -1 ، والتي يمس فيها منحنى الحركة ، الغلاف. وفقًا لذلك ، لذلك تنفصل نقاط التماس بفترة زمنية مقدارها نصف مدة الذبذبة.

3.5 Energy Consideration for Damped Harmonic Oscillator

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies

الطاقة الكلية للمتذبذب التوافقي المضمحل في اية لحظة هي مجموع للطاقات الحركية والكامنة

$$E_t = E_k + E_p$$

$E_t = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} k\, x^2 \dots\dots (1)$

To find ***time rate*** of change of E_t, we have to differentiate E_t with respect to t:

$$\frac{dE_t}{dt} = \frac{1}{2} 2m\dot{x}\frac{d\dot{x}}{dt} + \frac{1}{2} 2kx\frac{dx}{dt}$$

$$= m\ddot{x}\dot{x} + k\dot{x}x$$

$$\frac{dE_t}{dt} = (m\ddot{x} + kx)\dot{x} \dots\dots (2)$$

We have the Eq. of motion for the damped harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m\ddot{x} + kx = -c\dot{x} \dots\dots (3)$$

Sub. Eq. (3) in Eq. (2)

$$\therefore \frac{dE_t}{dt} = -c\dot{x}^2 \dots\dots (4)$$

This equation represents the rate at which the energy E_t dissipated as frictional heat by virtue of the viscous resistance to the motion.

هذه المعادلة تمثل معدل تبدد الطاقة الكلية الى حرارة بسبب الاحتكاك وهي مقدار سالب دائماً

3.6 Forced Harmonic Motion (Resonance)

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In this section, we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent.

Consider a damped harmonic oscillator motion affected by an external force (F_{ext}) that varying as a ***cosine*** wave with time t, the ***angular frequency*** w and ***amplitude*** (F_0) such that

هنا ندرس حركة المتذبذب التوافقي المضمحل المدفوع بقوة خارجية توافقية. اي قوة تتغير بدالة جيبية مع الزمن

$$F_{ext} = F_0 \cos(wt + \theta) \dots\dots (1)$$

$$F_{ext} = F_0\, e^{i(wt+\theta)} \dots\dots (2)$$

There are three forces attached on the body:

1. Elastic restoring Force $= -kx$
2. The viscous damping force $= -c\dot{x}$
3. External force $= F_{ext}$

هناك ثلاث قوى مؤثرة في الجسم:

- قوة معيدة مرنة $(-kx)$
- قوة لزوجة مضمحلة $(-c\dot{x})$
- قوة خارجية (F_{ext})

عليه تكون القوة الكلية المؤثرة على الجسم مجموع لهذه القوى الثلاث

So, total force is:

$$\therefore -kx - c\dot{x} + F_{ext} = m\ddot{x} \ \ldots\ldots (3)$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = F_0\ e^{i(wt+\theta)} \ \ldots\ldots (4)$$

Eq.(4) represent differential damped harmonic oscillator motion affected by an external force (F_{ext}). Suggested solution of this equation as:

معادلة (4) تمثل معادلة الحركة لمتذبذب توافقي مضمحل تحت تأثير قوة خارجية (F_{ext})

$$x = A\ e^{i(wt+\theta')} \ \ldots\ldots (5)$$

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}A\ e^{i(wt+\theta')}$$

$$\dot{x} = i\ A\ w\ e^{i(wt+\theta')} = iwx \quad \ldots\ldots (6)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}A\ e^{i(wt+\theta')}$$

$$\ddot{x} = i^2\ A\ w^2\ e^{i(wt+\theta')} = i^2w^2x = -w^2x \ \ldots\ldots (7)$$

Sub. Eqns.(5) ,(6) and (7) in Eq. (4).

$$-mAw^2e^{i(wt+\theta')} + cAwi\ e^{i(wt+\theta')} + kAe^{i(wt+\theta')} = F_0\ e^{i(wt+\theta)}\Big] * e^{-i(wt+\theta')}$$

$$\therefore\ -mA\,w^2 + iwcA + kA = F_0e^{i(wt+\theta)}.\,e^{-i\,(wt+\,\theta')}$$

$$-mAw^2 + iwcA + kA = F_0\left[e^{iwt}\ e^{i\theta}\ e^{-iwt}\ e^{-i\theta'}\ \right]$$

$$-mAw^2 + iwcA + kA = F_0e^{i(\theta-\theta')}$$

$$-mAw^2 + iwcA + kA = F_0[\cos(\theta-\theta') + i\sin(\theta-\theta')]$$

where $\varphi = (\theta - \theta') \equiv Phase\ difference\ (\boldsymbol{Phase\ angle})$

Separation between real and imaginary terms, we get:

$$-mA\,w^2 + kA = F_0\cos(\theta-\theta') = F_0\cos\varphi$$

$$iwcA = iF_0\sin(\theta-\theta) = i\ F_0\sin\varphi$$

$$A(k - mw^2) = F_0\cos\varphi \ \ldots\ldots (8)$$

$$cwA = F_0\sin\varphi \ \ldots\ldots (9)$$

Dividing Eq. (9) on Eq. (8)

$$\frac{cw}{k-mw^2} = \frac{F_0\sin\varphi}{F_0\cos\varphi} = \tan\varphi \ \ldots\ldots (10)$$

$$\frac{c}{2m} = \gamma \qquad \therefore \frac{c}{m} = 2\gamma$$
$$w_0 = \sqrt{\frac{k}{m}}$$

$$\therefore \tan\varphi = \frac{\frac{c}{m}w}{\frac{k}{m} - w^2}$$

$$\therefore \tan\varphi = \frac{2\gamma\, w}{{w_0}^2 - w^2} \quad \dots\dots (11)$$

Squaring and adding Eqns. (8) and (9)

$$A^2(k - mw^2)^2 + c^2w^2A^2 = F_0^2(\cos^2\varphi + \sin^2\varphi) = F_0^2 \quad \dots\dots (12)$$

$$A^2\left[(k - mw^2)^2 + c^2w^2\right] = F_0^2$$

$$A^2 = \frac{F_0^2}{(k - mw^2)^2 + c^2w^2}$$

$$\therefore A = \frac{F_0}{\sqrt{(k - mw^2)^2 + c^2w^2}} \quad \dots\dots (13)$$

by dividing the numerator and denominator on m

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{k}{m} - \frac{mw^2}{m}\right)^2 + \frac{c^2w^2}{m}}}$$

So, in term of γ and w_0

$$A = \frac{F_0/m}{\sqrt{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}} \quad \dots\dots (14)$$ ***Steady State Oscillation Amplitude***

Eq. (14) represent the amplitude (A) as a function of the driving frequency (w). The maximum value of amplitude valid only at ($w = w_0$) (***Resonance Frequency***). To find this frequency equal ***differential amplitude equation by zero.***

معادلة (14) تمثل السعة (A) كدالة للتردد الدافع (w). القيمة القصوى للسعة تتحقق فقط عند ($w = w_0$) (تردد الرنين).

$$\frac{dA}{dw} = \frac{d}{dw}\left[\frac{F_0/m}{\sqrt{\left(w_0^2 - w^2\right)^2 + 4\gamma^2w^2}}\right]$$

$$= \frac{F_0}{m}\frac{d}{dw}\left[(w_0^2 - w^2)^2 + 4\gamma^2w^2\right]^{-\frac{1}{2}}$$

$$= \frac{F_0}{m}\frac{d}{dw}\left[w_0^4 + w^4 - 2w_0^2\, w^2 + 4\gamma^2w^2\right]^{-\frac{1}{2}}$$

$$\frac{dA}{dw} = \frac{F_0}{m}\left(\frac{-1}{2}\right)[\, w_0^4 + w^4 - 2w_0^2\, w^2 + 4\gamma^2 w^2]^{-\frac{3}{2}} \cdot [0 + 4w^3 - 4w_0^2 w + 8\gamma^2 w]$$

$$= \frac{F_0}{m}\left(\frac{-1}{2}\right)\frac{4\,w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$= \frac{-F_0}{2m}\ \frac{4\,w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = \frac{-F_0}{2m}\ \frac{4\,w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}] * \frac{-m}{2F_0\, w}$$

$$\frac{dA}{dw} = \frac{mF_0}{4mF_0\, w}\ \frac{4w\left(w^2 - w_0^2 + 2\gamma^2\right)}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}} = \frac{\left(w^2 - w_0^2 + 2\gamma^2\right)}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = 0$$

$$\therefore w^2 - w_0^2 + 2\gamma^2 = 0$$

$$w^2 = w_0^2 - 2\gamma^2$$

$$w = w_r = (w_0^2 - 2\gamma^2)^{1/2} \ldots\ldots (15)$$ ***Resonant Frequency Equation***

where $w_r \equiv$ ***resonant frequency*** for ***maximum amplitude***.

In case of weak damping, that is, when $c \ll 2\sqrt{mk}$ $or\ \gamma \ll w_0$

Then $w_0 \simeq w_r$

From Eq. (14) and (15) we can find A_{max} in Resonant frequency.

$$w^2 = w_0^2 - 2\gamma^2 \ldots\ldots (16)$$

$$\therefore 2\gamma^2 = w_0^2 - w^2 \ldots\ldots (17)$$

Sub. Eq. (16) and (17) in Eq. (14)

$$A = \frac{F_0/m}{\sqrt{(2\gamma^2)^2 + 4\gamma^2(w_0^2 - 2\gamma^2)}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2 w_0^2 - 8\gamma^4}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^2 w_0^2 - 4\gamma^4}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^2(w_0^2-\gamma^2)}}$$

$$\therefore A_{max} = \frac{F_0/m}{2\gamma\sqrt{(w_0^2-\gamma^2)}} \dots \dots (18)$$

In other form:

$$A_{max} \simeq \frac{F_0}{2m\gamma\sqrt{(w_0^2-\gamma^2)}}$$

$$A_{max} \simeq \frac{F_0}{c\sqrt{(w_0^2-\gamma^2)}} \dots \dots (19)$$

$$\frac{c}{2m} = \gamma \quad \therefore c = 2m\gamma$$

In weak damping $\gamma \ll w_0$ then γ^2 can be ***neglect***

$$A_{max} \simeq \frac{F_0}{2\gamma m w_0} = \frac{F_0}{c w_0} \dots \dots (20)$$

$$F_0 \simeq 2A_{max}\gamma m w_0 = A_{max} c w_0$$

Sub. Eq. (20) in Eq. (14)

$$A = \frac{A_{max}\,\gamma}{\sqrt{(w_0-w)^2+\gamma^2}} \dots \dots (21)$$

when $|w_0 - w| = \gamma$

or $w = w_0 \mp \gamma$

Then

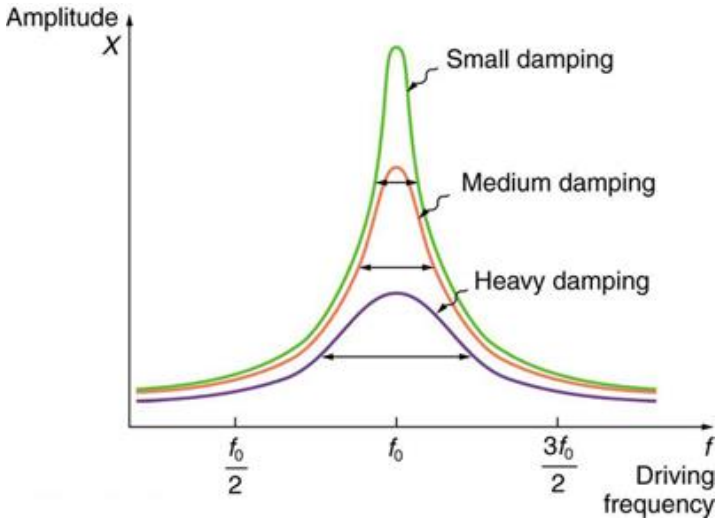
$$A = \frac{A_{max}\,\gamma}{\sqrt{(w_0-w_0+\gamma)^2+\gamma^2}}$$

$$A = \frac{A_{max}\,\gamma}{\sqrt{\gamma^2+\gamma^2}} = \frac{A_{max}\,\gamma}{\sqrt{2\gamma^2}} = \frac{A_{max}\,\gamma}{\sqrt{2}\,\gamma}$$

$$A^2 = \frac{1}{2}A^2{}_{max} \dots \dots (22)$$

This means that γ is a measure of the width of the resonance curve. Thus, 2γ is the frequency difference between the points for which the energy is down by a factor of from the energy at resonance because the energy is proportional to A^2.

هذا يعني أن γ هو مقياس لعرض منحنى الرنين. لذلك 2γ تمثل فرق التردد بين النقطتين اللتين تنخفض فيهما الطاقة بمقدار نصف طاقة الرنين لأن الطاقة تتناسب مع A^2



Another way of designating the sharpness of the resonance peak for the driven oscillator is in terms of the parameter (Q) called ***Quality Factor*** of the resonant system.

هناك طريقة أخرى لتعيين حدة قمة الرنين للمتبذب القسري وهي من خلال حساب المعامل (Q) الذي يسمى معامل النوعية للرنين.

$$Q = \frac{w_r}{2\gamma} \dots \dots (23)$$

In the case of weak damping

في حالة التضاؤل الضعيف

$$Q \approx \frac{w_0}{2\gamma} \dots \dots (24)$$

The total width Δw at the half energy points is approximately

$$\Delta w = 2\gamma \approx \frac{w_0}{Q} \dots \dots (25)$$

$$w = 2\pi f$$

$$\therefore \ \frac{\Delta w}{w_0} = \frac{\Delta f}{f_0} \simeq \frac{1}{Q} \dots \dots (26)$$

giving the fractional width of the resonance peak,

العرض الجزئي لقمة الرنين

$Q = 10^4$ [quartz oscillators]

Example:

Determine the resonance frequency and the quality factor for the damped oscillator if the damping frequency $= \frac{w_0}{4}$. Then find the phase angle θ if the applied frequency is $\frac{w_0}{2}$

Solution:

$$w_r = ({w_0}^2 - 2\gamma^2)^{1/2}$$

$$= ({w_0}^2 - \frac{2{w_0}^2}{16})^{1/2}$$

$$= w_0 \sqrt{\frac{7}{8}} = \sqrt{\frac{k}{m}} \sqrt{\frac{7}{8}}$$

$$Q = \frac{w_r}{2\gamma} = \frac{w_0\left(\frac{7}{8}\right)^{1/2}}{2\left(\frac{w_0}{4}\right)} = 2\sqrt{\frac{7}{8}} = 1.87$$

$$\tan\varphi = \frac{2\gamma\, w}{(w_0^2 - w^2)}$$

$$= \frac{2\,\frac{w_0}{4}\,\frac{w_0}{2}}{w_0^2 - (\frac{w_0}{2})^2} = \frac{2\,\frac{w_0^2}{8}}{w_0^2 - \frac{w_0^2}{4}}$$

$$= \frac{\frac{1}{4}w_0^2}{w_0^2\left(1 - \frac{1}{4}\right)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$

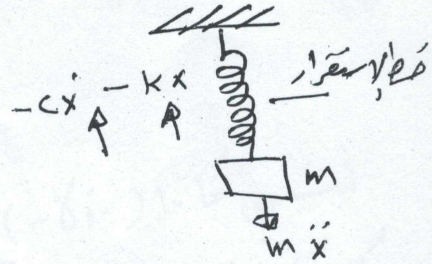
$$\tan\varphi = \frac{1}{3} \Rightarrow \ \varphi = \tan^{-1}\left(\frac{1}{3}\right) = 18.5^0$$

محاضرة 14

الحركة التوافقية المتضائلة

The Damped Harmonic Oscillation

نفرض ان هناك جسم كتلته m معلق بنابض ثابت النابض له k تؤثر عليه قوة معيقة (محققة) "قطعياً" مع سرعة الجسم (مقاومة الهواء)



من قانون نيوتن الثاني نحصل على معادلة رياضية للحركة

$$m\ddot{x} = -kx - c\dot{x}$$

$$m\ddot{x} + kx + c\dot{x} = 0 \quad \text{—(17)}$$

[معادلة الحركة للمتذبذب التوافقي المتضائل]

نستخدم الدالة الاسية كحل تجريبي اي نفرض ان

$$x = A e^{qt}$$

ثم نعوض في 17 فنحصل على

$$mq^2 + k + cq = 0$$

$$mq^2 + cq + k = 0$$

$ax^2 + bx + c = 0$

$x = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

$a = m$

$b = c$

$c = k$

من معادلة الدستور نحصل على

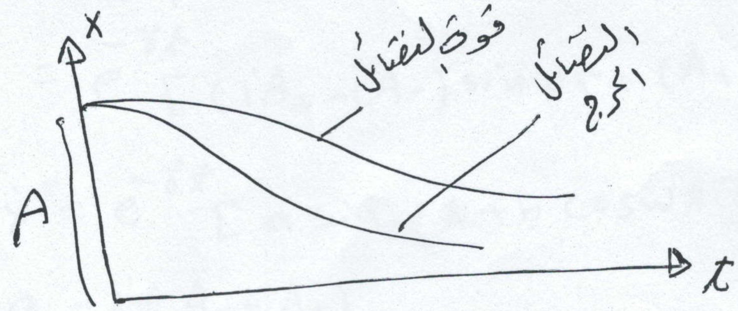
$$q = \frac{-c \pm (c^2 - 4mk)^{\frac{1}{2}}}{2m} \quad \text{—(18)}$$

من معادلة (18) نوضح الحالات التالية:

1.) $c^2 > 4mk$ (فوق التضائل over Damping)

2.) $c^2 = 4mk$ (التضائل الحرج Critical Damping)

حيث هناك الحالتين قيمة q حقيقية وسالبة اي ان
الازاحة x تهبط اسياً" مع الزمن . اي الحالتين
متذبذبة وكما يلي



لنسمي كاما $(-\gamma_1)$ و $(-\gamma_2)$ قيمتان لـ q من المعادلة (18) عندئذ
يمكن كتابة الحل العام للمعادلة (17) بالشكل التالي

$$x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t} \longrightarrow \text{فوق التضاؤل}$$

في حالة التضاؤل الحرج تكون هناك جذر واحدة لـ q اي

$$\gamma_1 = \gamma_2 = \gamma = -c/2m$$

فيكون الحل العام للتضاؤل الحرج

$$x = e^{-\gamma t}[A_1 + A_2 t] \longrightarrow \text{التضاؤل الحرج}$$

اذا كانت ثابت المقاومة c من الصغر بحيث

$$c^2 < 4mk \longrightarrow$$ (دون التضاؤل under Damping)

وفي هذه الحالة q خيالية والحل العام هو

$$x = A_+ e^{(-\gamma + i\omega_1)t} + A_- e^{(-\gamma - i\omega)t}$$

حيث ان

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2}$$

$$e^{iu} = \cos u + i \sin u$$

نحصل على

$$X = e^{-\gamma t}\left[A_+ e^{i\omega t} + A_- e^{-i\omega t}\right]$$

$$= e^{-\gamma t}\left[(iA_+ - iA_-)\sin\omega t + (A_+ + A_-)\cos\omega t\right]$$

or

$$x = e^{-\gamma t}\left[a \sin\omega t + b\cos\omega t\right]$$

حيث ان

$$a = i(A_+ - A_-)$$

$$b = (A_+ + A_-)$$

كذلك يمكن كتابة الحل على النحو التالي

$$X = A e^{-\gamma t}\cos(\omega t + \theta_0) \quad \text{——} (19)$$

حيث ان

$$A = \sqrt{a^2 + b^2}$$

$$\theta = -\tan^{-1}\left(\frac{a}{b}\right)$$

من معادلة (19) نلاحظ ان صيغة الحل عبارة عن حركة اهتزازية ذات سعة ($Ae^{-\gamma t}$) متناقصة اسياً مع الزمن كما ان التردد (ω_1) اقل من التردد الحر الغير متضائل (ω_0) اذا كانت (γ) صغيرة فان

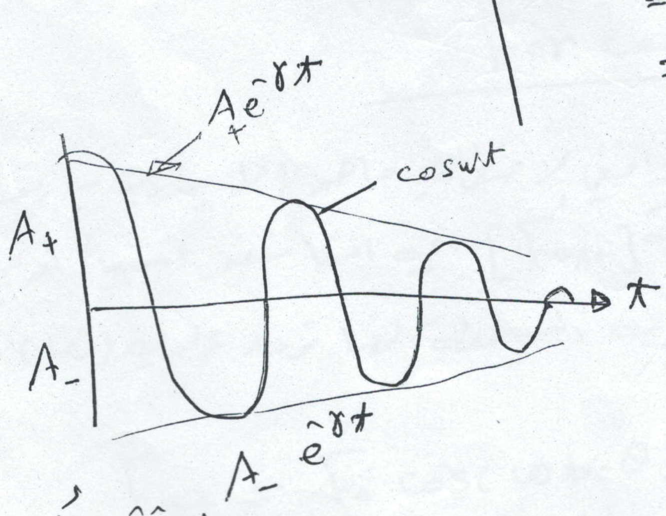
$$\omega = \omega_0 - \frac{\gamma^2}{2\omega_0}$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

$$= (\omega_0^2 - \gamma^2)^{\frac{1}{2}}$$

$$= \omega_0\left(1 - \frac{\gamma^2}{\omega_0^2}\right)^{\frac{1}{2}}$$

$$= \omega_0 - \frac{\gamma^2}{2\omega_0}$$



ان الطاقة الكلية للمتذبذب التوافقي الحر في اي لحظة تساوي مجموع الطاقة الحركية $\frac{1}{2}m\dot{x}^2$ و $\frac{1}{2}kx^2$ الكامنة ، اي ان

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

تكون ثابتة للمتذبذب الحر [الغير المتضائل] $= E$

اما للمتذبذب التوافقي المتضائل فان E غير ثابتة اي ان

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + k\dot{x}x$$
$$= \dot{x}(m\ddot{x} + kx)$$

بما ان المتذبذب التوافقي المتضائل

$$m\ddot{x} + kx = -c\dot{x}$$ from eq (17)

$$\frac{dE}{dt} = -c\dot{x}^2$$ المعدل الزمني لتغير الطاقة الكلية للمتذبذب التوافقي المتضائل

الاشارة السالبة تعني ان الطاقة المتبددة بالاحتكاك وتتحول الى حرارة

* المتذبذب التوافقي القسري (الاجباري) .

<u>For ceed Harmonic oscilation</u>

~~نفرض~~ . ان المتذبذب التوافقي الاجباري ماهو الا متذبذب توافقي متضائل مدفوع بقوة خارجية [F_{ext}] اي انها تتغير اسياً او جيبياً .
نفرض ان هذه القوة الخارجية المسلطة لها تردد زاوي (ω) وسعة قدرها (F_0) اي ان

$$F_{ext} = F_0 \cos(\omega t + \theta) \quad \text{—— 20}$$

$$F_{ext} = F_0 e^{i(\omega t+\theta)}$$

اذن معادلة الحركة المتذبذب التوافقي القسري هي

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i(\omega t+\theta)}$$

سنجرب الحل التجريبي التالي

$$x = A e^{i(\omega t+\delta)}$$

لنعوض الحل في المعادلة التفاضلية للحركة فنحصل على

$$-m\omega^2 A + i\omega c A + kA = F_0 e^{i(\theta-\delta)}$$

$$= F_0[\cos(\theta-\delta) + i\sin(\theta-\delta)]$$

بعد مساواة الحدود الحقيقية والتخيلية في المعادلة نحصل على

$$A(k - m\omega^2) = F_0 \cos(\phi) \quad \longrightarrow (21)$$

$$\omega c A = F_0 \sin(\phi) \quad \longrightarrow (22)$$

$$\phi = \theta - \delta \quad \text{حيث}$$

تقسيم هاتين المعادلتين فنحصل على

$$\tan(\phi) = \frac{c\omega}{(k - m\omega^2)} = \frac{2\gamma\omega}{\omega_0 - \omega^2} \quad \longrightarrow (23)$$

$\gamma = \frac{c}{2m}$

$\omega_0 = \frac{k}{m}$

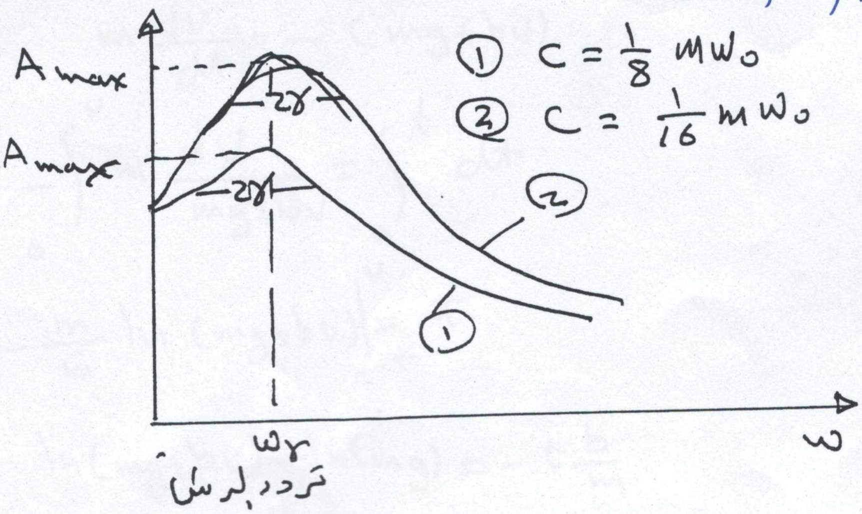
نربع هاتين المعادلتين (21) و (22) وثم نجمعهما فنحصل على

$$A^2(k - m\omega^2)^2 + c^2\omega^2 A^2 = F_0^2 \qquad \sin^2\phi + \cos^2\phi = 1$$

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

المعادلة المذكورة تمثل العلاقة بين السعة A و التردد للدافع المؤثر ω . من الرسم هناك قيمة عظمى لـ A عند تردد معلوم ω_R والذي يسمى تردد الرنين .

الى هنا وصلنا
10/11/2017م



لايجاد تردد الرنين ω_r نشتق السعة A بالنسبة لـ ω التردد ونضع الناتج مساوي للصفر اي ان

$$\frac{dA}{d\omega} = 0$$

فنحصل على

$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2} \quad \text{——} \ (24) \quad H.W$$

اذا كانت γ صغيرة فان التردد الرنيني ω_r يصبح كما يلي

$$\omega_r \simeq \omega_0 - \frac{\gamma^2}{\omega_0} \quad \text{——} \ (25)$$

اسئلة نهاية الفصل الاول

س/ برهن ان المعدل الزمني لتغير الطاقة الكلية E للمتذبذب التوافقي المتضائل تعطى بالعلاقة التالية $\frac{dE}{dt} = -cx^2$ حيث c مقدار ثابت وماذا تعني الاشارة السالبة؟

س/ جسم كتلته m في اللحظة $t=0$ سقط من السكون بصورة شاقولية في مائع ما فاذا كانت قوة اعاقة المائع له هي $F = -bv$ حيث b مقدار ثابت و v سرعة الجسم الساقط.
برهن ان حركته بعد مرور فترة زمنية مقدارها τ تعطى بالعلاقة التالية

$$v = \frac{m}{b} g \left[e^{-\frac{b}{m} t} - 1 \right]$$

الحل

$$F = ma = -mg - bV$$

$$m\frac{dv}{dt} = -(mg + bv)$$

$$-\int_0^v m\frac{dv}{mg+bv} = \int_0^t dt$$

$$-\frac{m}{b}\ln(mg+bv)\Big|_0^v = t$$

$$\ln(mg+bv) - \ln(mg) = -t\frac{b}{m}$$

$$\frac{mg+bv}{mg} = e^{-t\frac{b}{m}}$$

$$1 + \frac{bv}{mg} = e^{-t\frac{b}{m}}$$

$$V = \frac{mg}{b}\left[e^{-\frac{b}{m}t} - 1\right] \quad //$$

س٣/ اذا كانت السعة للمتذبذب التوافقي القسري تعطى بالعلاقة التالية

$$A = \frac{F_o/m}{\sqrt{(\omega_o^2-\omega^2)^2 + 4\gamma^2\omega^2}}$$

برهن ان تردد الرنين

$$\omega_r = \sqrt{\omega_o^2 - 2\gamma^2}$$

حيث F_o سعة القوة القسرية

m كتلة الجسم المتذبذب

ω_o التردد الحر

γ مقدار ثابت

الحل

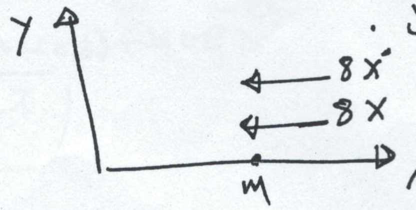
$$\frac{dA}{d\omega} = -\frac{\frac{F_o}{m}\frac{1}{2}\left[(\omega_o^2-\omega^2)^2+4\gamma^2\omega^2\right]^{-\frac{1}{2}}}{(\omega_o^2-\omega^2)^2+4\gamma^2\omega^2} * \left[2(\omega_o^2-\omega^2)(-2\omega) + 8\gamma^2\omega\right]$$

$$\frac{dA}{d\omega} = 0 \Rightarrow -F_o\left[(\omega_o^2-\omega^2)^2(-2\omega) + 8\omega\gamma^2\right] = 0$$

$$2\gamma^2 - (\omega_o^2 - \omega_r^2) = 0$$

$$\omega_r = \sqrt{\omega_o^2 - 2\gamma^2} \quad //$$

س / جسم كتلته m تساوي 2 gm يتحرك على الاحداثي X بسبب تؤثر عليه قوة جذب مقدارها $8X$ وقوة معيقة مقدارها $8\dot{X}$ كما في الشكل.



في اللحظة كان الجسم في الموضع $X_0 = 20\ cm$ وسرعته $V = 0$.

1- اي نوع من التضاؤل لحركة الجسم. برهن ان هذا النوع من التضاؤل هو تضاؤل حرج.

2- برهن ان موضع الجسم بعد مرور فترة زمنية مقدارها t هو

$$X = 20 e^{-2t}(1+2t)$$

3- برهن ان سرعة الجسم بعد مرور فترة زمنية مقدارها t هي

$$V = -80 t e^{-2t}$$

علماً بأن الحل العام للمعادلة التفاضلية للتضاؤل الحرج هو

$$X = e^{-\gamma t}(A+Bt)$$

حيث A و B و γ ثوابت حيث $\gamma = \frac{c}{2m}$ و c ثابت.

الحل

(1)

$$c^2 = (8)^2 = 64$$

$$4mk = 4*2*8 = 64$$

$$c^2 = 4mk$$

تضاؤل حرج

(2)

$$\gamma = \frac{c}{2m} = \frac{8}{4} = 2$$

$$X = e^{-2t}(A+Bt)$$

at $t = 0$

$x = 20\ cm$

$$\boxed{A = 20}$$

$$\frac{dx}{dt} = -2e^{-2t}(A+Bt) + Be^{-2t} = V$$

$$= -2(A) + B = 0$$

$$\therefore \boxed{x = 20e^{-2t}[1+2t]}$$

$$V = \frac{dx}{dt} = -2e^{-2t}(A+Bt) + Be^{-2t}$$

$$= -2e^{-2t}(20+40t) + 40e^{-2t}$$

$$\boxed{V = -80te^{-2t}}$$

س/ زورق يسير بسرعة ابتدائية مقدارها v_0 اطفئ محركه في اللحظة $t=0$ في الموضع $x_0=0$ بعدها اصبح يتحرك بتعجيل تباطؤي يتناسب طرديا" مع مربع سرعته $a=-cV^2$ حيث c مقدار ثابت

١- برهن ان سرعته V بعد مرور فترة زمنية t تعطى بالعلاقة التالية

$$V = \frac{V_0}{1+V_0ct}$$

٢- برهن ان سرعته بعد مسافة قدرها x تعطى بالعلاقة التالية

$$V = V_0e^{-cx}$$

الحل

(1)

$$a = -cV^2$$

$$\frac{dv}{dt} = -cV^2$$

$$\int_{v_0}^{v}\frac{dv}{V^2} = -c\int_0^t dt$$

$$-\frac{1}{V}\Big|_{v_0}^{v} = -ct$$

$$-\frac{1}{V}+\frac{1}{V_0} = -ct$$

$$\frac{1}{V}-\frac{1}{V_0} = ct$$

$$\frac{1}{V} = ct+\frac{1}{V_0}$$

$$= \frac{ctV_0+1}{V_0}$$

$$\boxed{V = \frac{V_0}{1+ctV_0}}$$

(2)

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$

$$= v\frac{dv}{dx}$$

$$-cV^2 = v\frac{dv}{dx}$$

$$-cV = \frac{dv}{dx}$$

$$\int_{v_0}^{v}\frac{dv}{V} = -\int_{x_0=0}^{x}c\,dx$$

$$\ln V\Big|_{v_0}^{v} = -cx$$

$$\ln v - \ln v_0 = -cx$$

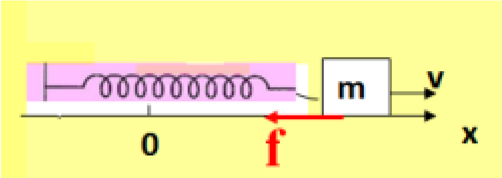
$$\ln\frac{V}{V_0} = -cx$$

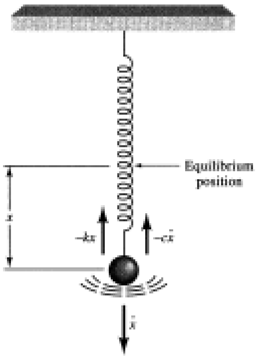
$$\frac{V}{V_0} = e^{-cx}$$

$$\boxed{V = V_0e^{-cx}}$$

//

Damped harmonic motion





Up to this point we have assumed that no frictional force act on the system.

For real oscillator, there may be friction, air resistance act on the system, the amplitude will decrease.

This loss in amplitude is called "*damping*" and the motion is called "damped harmonic motion".

Friction or other sources of external work can lead to a **loss of energy,** (known as dissipation), from an oscillating system. This phenomenon is referred to as **damping**.

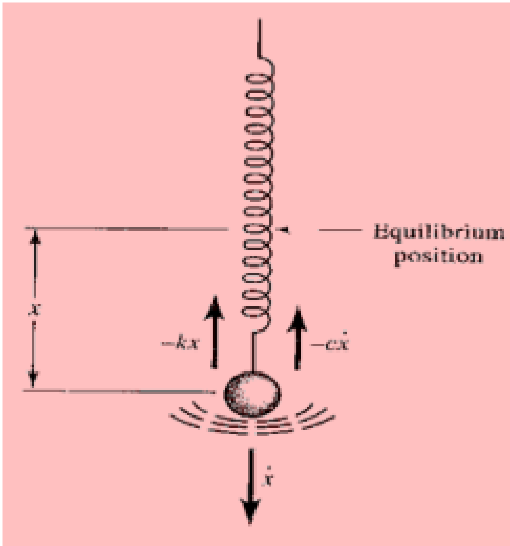
Damping has two principal effects on the oscillating system. It

- **decreases the amplitude of the oscillations and**
- **decreases the frequency (increases the period) of oscillations.**

Damped Harmonic Motion

Consider an object of mass m is supported by a light spring of constant k. We assume that there is a viscous retarding force (-cv) that is a linear function of the velocity.

The differential equation of motion is, therefore,



$$m\ddot{x} + c\dot{x} + kx = 0$$

Or,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

If we introduce the damping factor (γ) defined as

$$\gamma = \frac{c}{2m}$$

Then

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

Simple sine or cosine solutions do not work, because of the presence of the velocity-dependent term.

The suitable solution for this case is;

$$x(t) = A_1 e^{-(\gamma - q)t} + A_2 e^{-(\gamma + q)t}$$

Let D be the differential operator d/dt.

$$\left[D^2 + 2\gamma D + \omega_0^2\right]x = 0$$

$$\left[D + \gamma - \sqrt{\gamma^2 - \omega_0^2}\right]\left[D + \gamma + \sqrt{\gamma^2 - \omega_0^2}\right]x = 0$$

where

$$q = \sqrt{\gamma^2 - \omega_0^2}$$

There are three possible situations:

I. q real > 0 (**Overdamping**)
II. q real = 0 (**Critical damping**)
III. q imaginary (**Underdamping**)

I. Overdamped case:

Both exponents are real. The constants A1 and A2 are determined by the initial conditions. The motion is an exponential decay with two different decay constants, $(\gamma - q)$ and $(\gamma + q)$. **The mass will be prevented from oscillating by the strong damping force.**

II. Critical damping case:

Here $q = 0$. The two exponents are each equal to γ.

$$(D + \gamma)(D + \gamma)x = 0$$

we make the substitution $u = (D + \gamma)x$, which gives

$$(D + \gamma)u = 0$$

$$u = Ae^{-\gamma t}$$

Equating this to $(D + \gamma)x$, the final solution is obtained as follows:

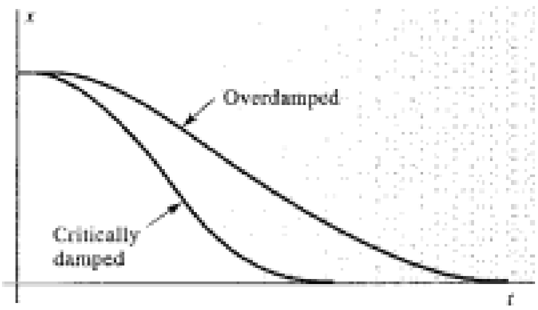
$$Ae^{-\gamma t} = (D + \gamma)x$$

$$A = e^{\gamma t}(D + \gamma)x = D(xe^{\gamma t})$$

$$\therefore\ xe^{\gamma t} = At + B$$

$$x(t) = Ate^{-\gamma t} + Be^{-\gamma t}$$

As in case I, **the motion is a returning to equilibrium with no oscillation.**



EXAMPLE 3.4.1

An automobile suspension system is critically damped, and its period of free oscillation with no damping is 1 s. If the system is initially displaced by an amount x_0 and released with zero initial velocity, find the displacement at $t = 1$ s.

Solution:

For critical damping we have $\gamma = c/2m = (k/m)^{1/2} = \omega_0 = 2\pi/T_0$. Hence, $\gamma = 2\pi\ \text{s}^{-1}$ in our case, because $T_0 = 1$ s. Now the general expression for the displacement in the critically damped case $x(t) = (At + B)e^{-\gamma t}$, so, for $t = 0$, $x_0 = B$. Differentiating, we have $\dot{x}(t) = (A - \gamma B - \gamma A t)e^{-\gamma t}$, which gives $\dot{x}_0 = A - \gamma B = 0$, so $A = \gamma B = \gamma x_0$ in our problem. Accordingly,

$$x(t) = x_0(1 + \gamma t)e^{-\gamma t} = x_0(1 + 2\pi t)e^{-2\pi t}$$

is the displacement as a function of time. For $t = 1$ s, we obtain

$$x_0(1 + 2\pi)e^{-2\pi} = x_0(7.28)e^{-6.28} = 0.0136\, x_0$$

The system has practically returned to equilibrium.

III. Underdamping case:

If the constant γ is small enough that q is **imaginary. The motion, in this case, is oscillatory but with an ultimate death.** Let introduce the constant ω_d such that; $q = i\omega_d$

Then;

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

Which is known as the **angular frequency of the under-damped oscillator.** The solution for the underdamped oscillator could be;

$$x(t) = e^{-\gamma t}\left(\frac{A}{2}e^{+i(\omega_d t + \theta_0)} + \frac{A}{2}e^{-i(\omega_d t + \theta_0)}\right)$$

We now apply Euler's identity[3] to the above expressions, thus obtaining

$$\frac{A}{2}e^{+i(\omega_d t + \theta_0)} = \frac{A}{2}\cos(\omega_d t + \theta_0) + i\frac{A}{2}\sin(\omega_d t + \theta_0)$$

$$\frac{A}{2}e^{-i(\omega_d t + \theta_0)} = \frac{A}{2}\cos(\omega_d t + \theta_0) - i\frac{A}{2}\sin(\omega_d t + \theta_0)$$

$$\therefore x(t) = e^{-\gamma t}(A\cos(\omega_d t + \theta_0))$$

we can express the solution equally well as a sine function:

$$x(t) = e^{-\gamma t}(A\sin(\omega_d t + \phi_0))$$

The main differences between the solution of the underdamped oscillator and the undamped oscillator are :

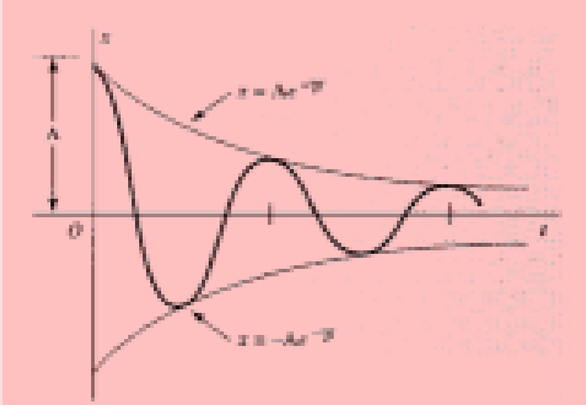
1- The presence of the real exponential factor $e^{-\gamma t}$ leads to a gradual death of the oscillations.

2- The underdamped oscillator vibrates a little more **slowly** than the undamped oscillator does. I.e, $\omega_d < \omega_0$ because of the presence of the damping force.

The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$$

Thus, in **one complete period** the amplitude diminishes by a factor $e^{-\gamma T_d}$



EXAMPLE 3.4.2

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

Solution:

We have $\omega_d = \frac{1}{2}\omega_0 = (\omega_0^2 - \gamma^2)^{1/2}$, which gives $\omega_0^2/4 = \omega_0^2 - \gamma^2$, so $\gamma = \omega_0(3/4)^{1/2}$. Consequently,

$$\gamma T_d = \omega_0(3/4)^{1/2}\,[2\pi/(\omega_0/2)] = 10.88$$

Thus, the amplitude ratio is

$$e^{-\gamma T_d} = e^{-10.88} = 0.00002$$

This is a *highly damped* oscillator.

Energy Considerations

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \qquad (3.4.18)$$

This is constant for the undamped oscillator, as stated previously. Let us differentiate the above expression with respect to t:

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} \qquad (3.4.19)$$

Now the differential equation of motion is $m\ddot{x} + c\dot{x} + kx = 0$, or $m\ddot{x} + kx = -c\dot{x}$. Thus, we can write

$$\frac{dE}{dt} = -c\dot{x}^2 \qquad (3.4.20)$$

Quality Factor

$$Quality\ Factor(Q) = \frac{2\pi\ times\ the\ energy\ stored\ in\ the\ oscillator}{the\ energy\ lost\ in\ a\ single\ period\ of\ oscillation\ T_d} = \frac{2\pi\ E}{(\Delta E)_{T_d}}$$

If the oscillator is weakly damped, the energy lost per cycle is small and Q is large.

The ratio of the energy stored in the oscillator to that lost in a single period of oscillation is characterized by a parameter Q, called **the quality factor**. This factor is related to w_d by the relation

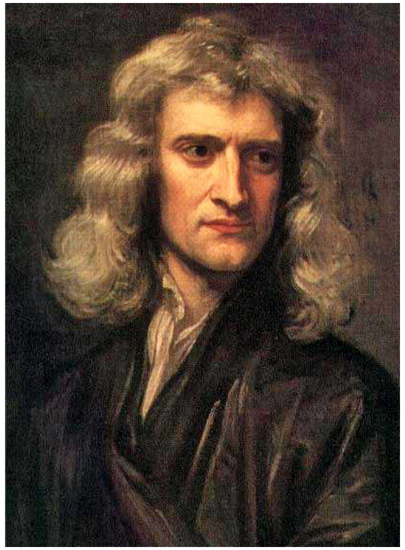
$$Q = \frac{\omega_d}{2\gamma}$$

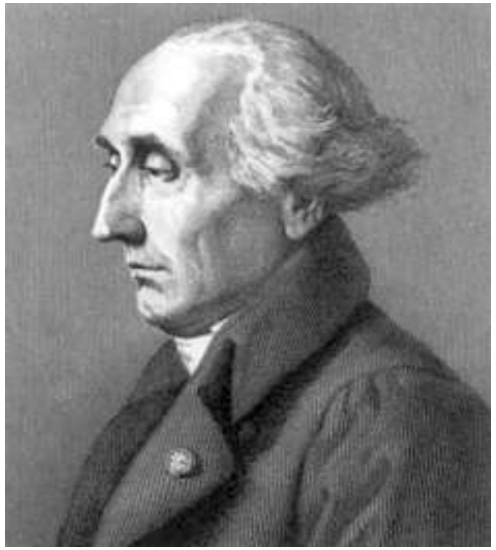
Chapter 7

محاضرة 15

Hamilton's Principle- Lagrangian and Hamiltonian Dynamics

Sir Isaac Newton

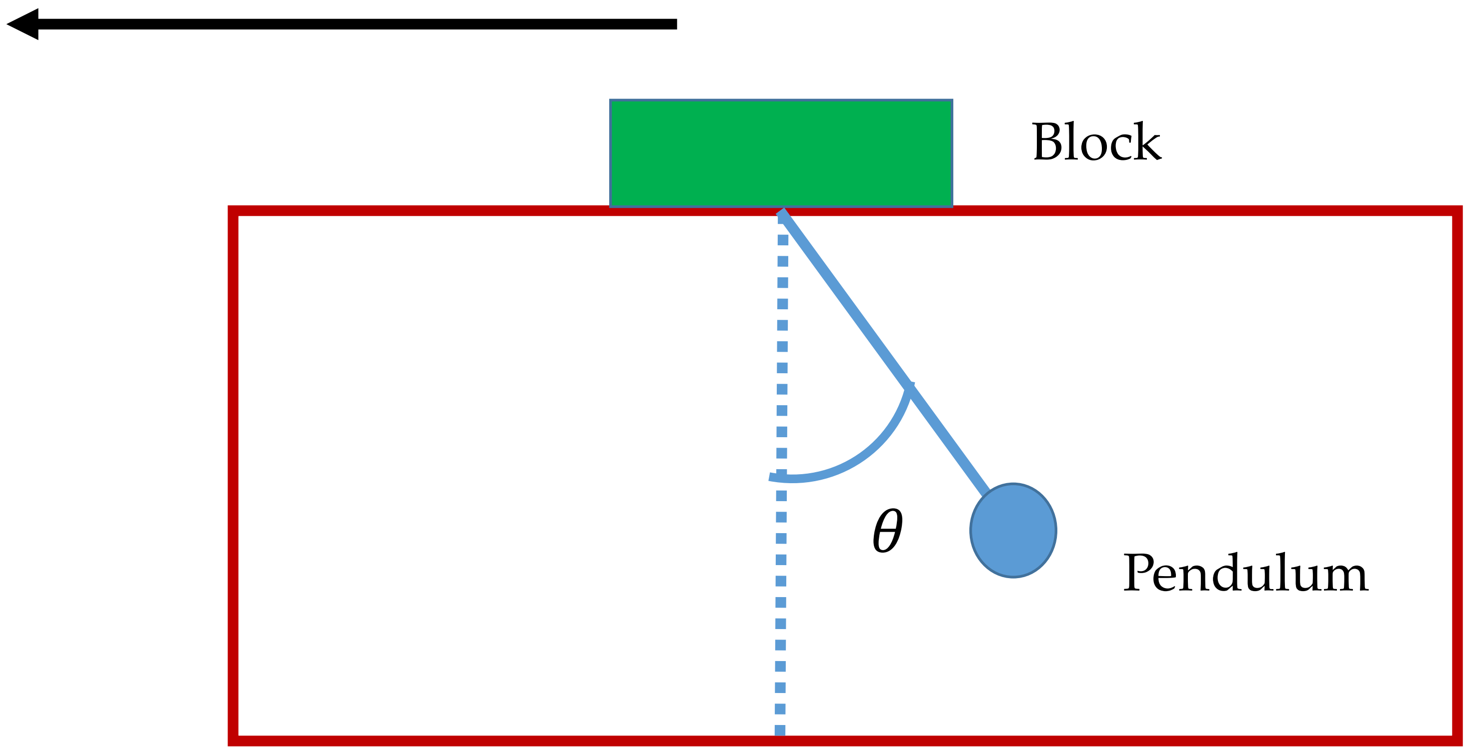




Joseph-Louis Lagrange

Experience has shown that a particle's motion in an internal reference frame is correctly described by the Newtonian equation (see Chapter 2) $F = \dot{p}$

This is a complicated system!



Lagrangian's equations (L)

$$L = \underbrace{T}_{\text{Kinetic energy}} - \underbrace{U}_{\text{Potential energy}}$$

Newtonian equations

Lagrangian's equations

Position
Velocity
Acceleration

Equation of motion

Energy

The Euler-Lagrange equations is given by

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1,2,3$$

2. Simple Harmonic Oscillator

تذكير من الفصل الثالث

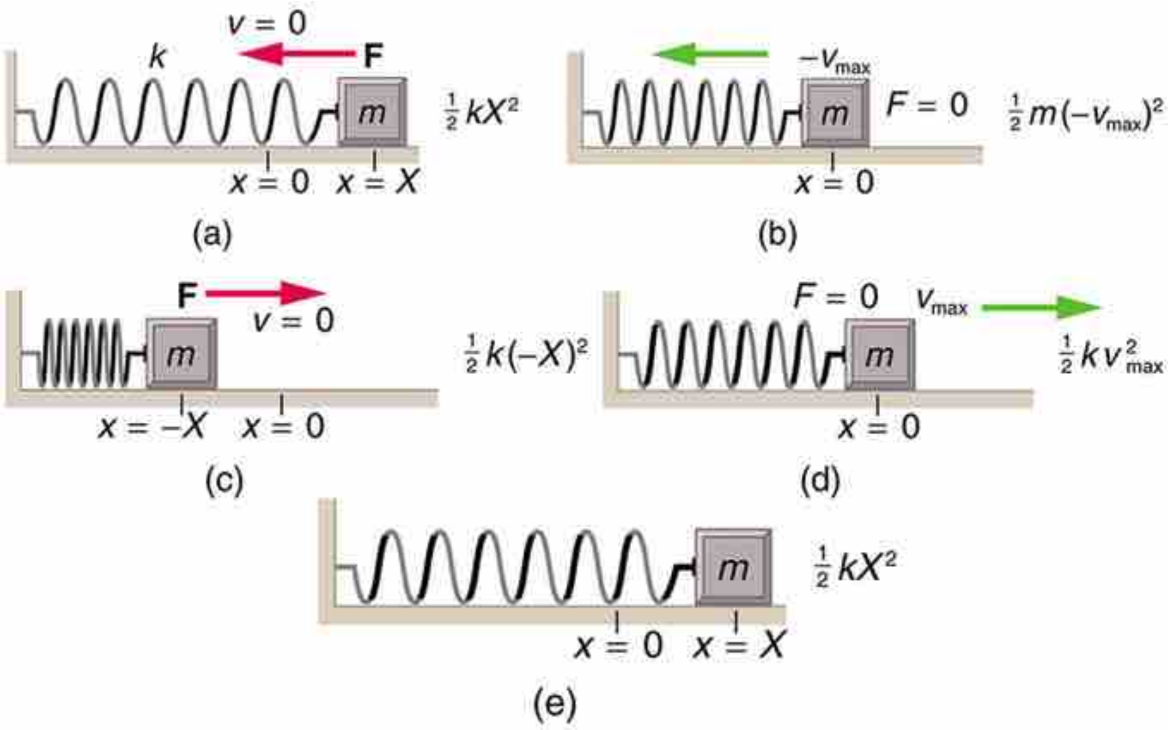
The equation of motion

Substitute the Hooke's Law in Newtonian equation

$$-kx = F$$

$$-kx = m\ddot{x}$$

Let $\omega_0^2 = {}^{k}/_{m}$



$$m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \left(\frac{k}{m}\right)x = 0 \Rightarrow \boxed{\ddot{x} + \omega_0^2 x = 0}$$

Example 7.1: Use the Lagrange equation to obtain the equation of motion for one-dimensional harmonic oscillator.

Answer:

With the usual expressions for the kinetic and potential energies, we have

$$L = T - U$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\frac{\partial L}{\partial x} = -kx$$

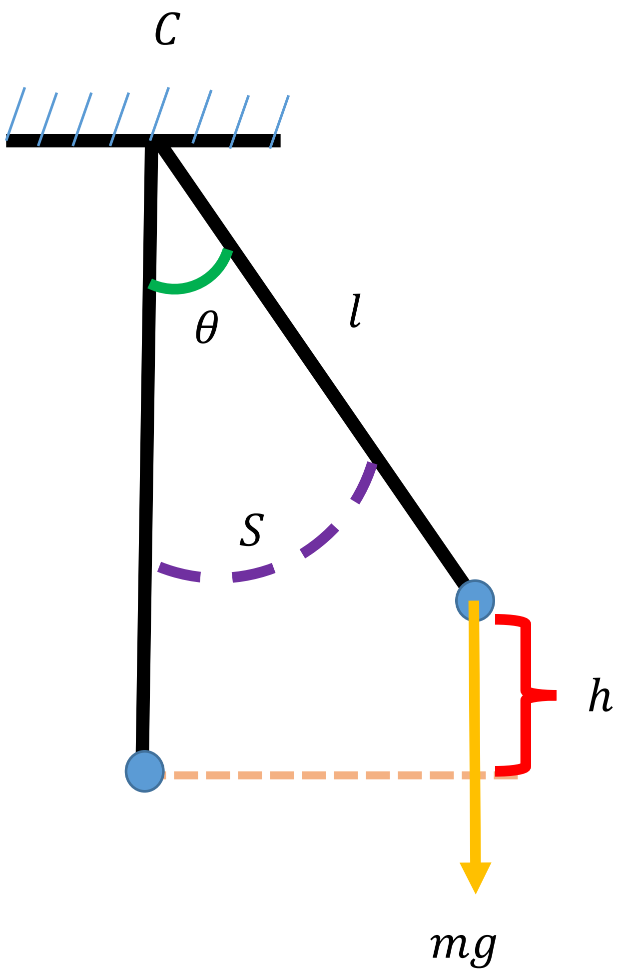
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

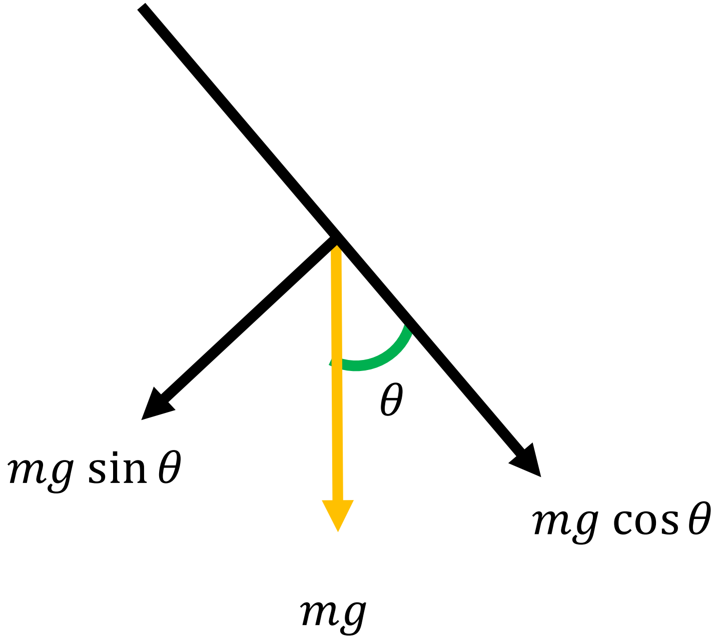
$$m\ddot{x} + kx = 0$$

This is identical with the equation of motion obtained using Newtonian mechanics (See Chapter 3).

3. The Simple Pendulum

تذكير من الفصل الثالث





$$F_s = -mg\sin\theta \quad \Rightarrow \quad m\ddot{s} = -mg\sin\theta \quad \Rightarrow \quad \cancel{m}\ddot{s} + \cancel{m}g\sin\theta = 0$$

$$\ddot{s} + g\sin\theta = 0 \qquad s = l\theta \qquad \ddot{s} = l\ddot{\theta}$$

$$l\ddot{\theta} + g\sin\theta = 0 \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0} \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

$$\boxed{\tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}}$$

$$\boxed{\omega_0 = \sqrt{\frac{g}{l}}}$$

Simple pendulum

Example 7.2: Use the Lagrange equation to obtain the equation of motion of Simple pendulum.

Answer:

The kinetic and potential energies of the system are given by:

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

$$L = T - U$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial \theta} = -mgl\, sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2 \ddot{\theta}$$

$$-mgl\, sin\theta - ml^2 \ddot{\theta} = 0$$

$$g\, sin\theta + l\ddot{\theta} = 0$$

$$\ddot{\theta} + \left(\frac{g}{l}\right) sin\theta = 0$$

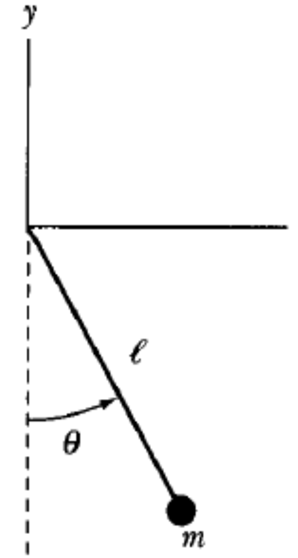
This is a remarkable result!

It has been obtained by calculating the kinetic and potential energies in terms of θ rather than x and then applying a set of operations designed for use with rectangular rather than angular coordinates.

Another important characteristic of the method used in two preceding simple examples is that nowhere in the calculation did there enter any statement **regarding force.**

Example 7.3. Use the (x, y) coordinate system to find the kinetic energy T, potential energy U, and the Lagrangian L for a simple pendulum (length l, mass bob m) moving in x,y plane .Determine the transformation equations from the (x, y) rectangular system to the coordinate θ. Find the equation of motion.

Answer:



The kinetic and potential energies and the Lagrangian become

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad\qquad U = mgy$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

Inspection reveals that the motion can be better described by using θ and $\dot{\theta}$. Let's transform x and y into the coordinate θ and then find L in terms of θ.

$$x = l\sin\theta \qquad\qquad y = -l\cos\theta$$

We now find for $\dot{x}$ and $\dot{y}$

$$\dot{x} = l\,\dot{\theta}\cos\theta \qquad\qquad \dot{y} = l\,\dot{\theta}\sin\theta$$

$$L = \frac{m}{2}\left(l^2\dot{\theta}^2 cos^2\theta + l^2\dot{\theta}^2 sin^2\theta\right) + mgl\cos\theta$$

$$L = \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$\ddot{\theta} + (\frac{g}{l})\ sin\theta = 0$$

7.4 Lagrange's Equations of motion in Generalized coordinates.

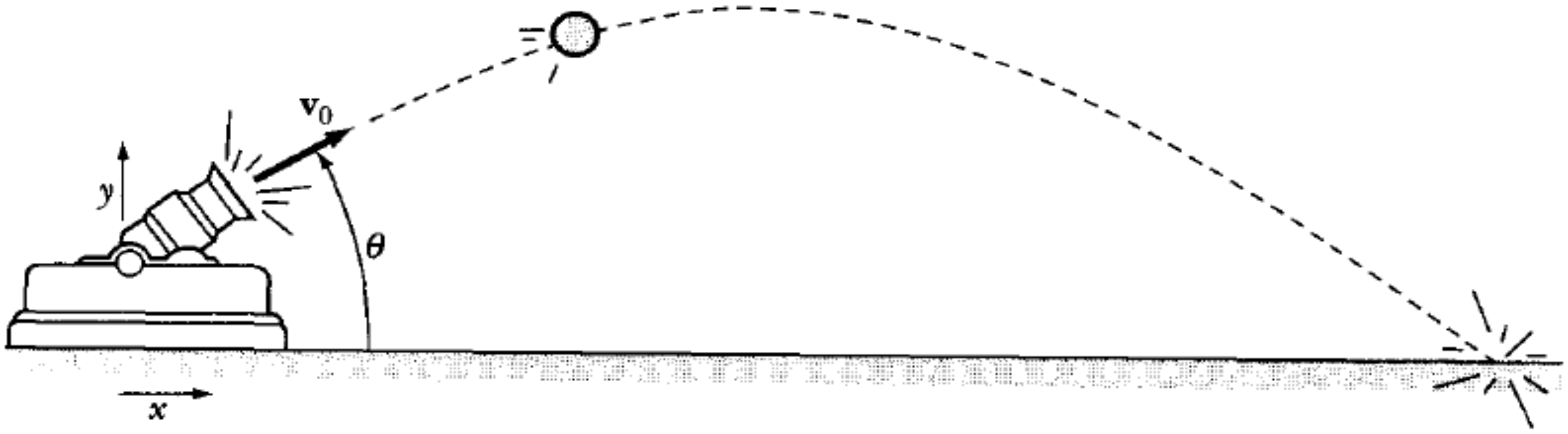
$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1,2, \dots \dots \dots .., s$$

It is important to realize that the validity of Lagrange's equation requires the following two conditions:

1. The force acting on the system (apart from any forces of constraint) must be derivable from the potential
2. The equations of constraint must be relations that connect the coordinates of the particles and may be functions of the time.

محاضرة 16

Example 7.4: Consider the case of projectile motion under gravity in two dimensions (as was discussed in Chapter 2). Find the equations of motion in both Cartesian and polar coordinates.



In Cartesian coordinate, we use xx (horizantoal) and y y (vertical).

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = mgy$$

Where $U = 0 \; at \; y = 0$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

x:	$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$ $$0 - \frac{d}{dt}\, m\dot{x} = 0$$ $$\ddot{x} = 0$$
y:	$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$ $$-mg - \frac{d}{dt}(m\dot{y}) = 0$$ $$\ddot{y} = -g$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

In polar coordinate, we use r (in radial direction) and θ (elevation angle from horizontal)

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\theta}\right)^2 \qquad U = mgr\sin\theta$$

Where $U = 0$ for $\theta = 0$

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

r:	$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$ $mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$ $r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$
θ:	$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$ $-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$ $-gr\cos\theta - 2r\dot{r}\theta - r^2\ddot{\theta} = 0$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

Canonical Equations of Motion-Hamiltonian

In the previous section, we found that if the potential energy of a system is velocity independent, then the linear momentum components in rectangular coordinates are given by

$$p_i = \frac{\partial L}{\partial \dot{x}}$$

By analogy, we extend this result to the case in which the Lagrangian is expressed in generalized coordinates and define the generalized momenta* according to

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Using the definition of the generalized momenta for the Hamiltonian may be written as

$$H = \sum p_j \dot{q}_j - L$$

$\dot{q}_j = \dot{x} = v$

$p_j = p = linear\ momentum$

$$H = \sum p_j \, \dot{x}_j - L$$

$$As\ T = \left(\frac{1}{2} m v^2\right)$$

$$sine\ \ pv = (mv)\, v = mv^2$$

$$2\left(\frac{1}{2} m v^2\right)$$

$$2\,T$$

$$H = 2T - T + U$$

$$H = T + U$$

$$L = T - U$$

Hamiltonian

Lagrangian

Example 7.5: Find the equations of motion for a system of particle moving in a potential region where $U = cx$ using Hamiltonian method.

If $U = cx$

Particle gain height

$$H = T + U$$

$$H = \frac{1}{2}m\dot{x}^2 + cx$$

$sine\ p = m\dot{x}$

$$p^2 = m^2\dot{x}^2$$

$$H = \frac{m^2\dot{x}^2}{2m} + cx$$

$$H = \frac{p^2}{2m} + cx$$

$$\frac{\partial H}{\partial x} = c$$

$$\frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\frac{\partial H}{\partial p} = \frac{m\dot{x}}{m} = \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}$$

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right)+\frac{\partial H}{\partial x}=0$$

$$m\ddot{x}+c=0$$

OR

$$\ddot{x}=-\frac{c}{m}$$

$Equation\ of\ motion$

Example 7.6: Find the equations of motion for a system of simple Oscillator using the Hamiltonian method.

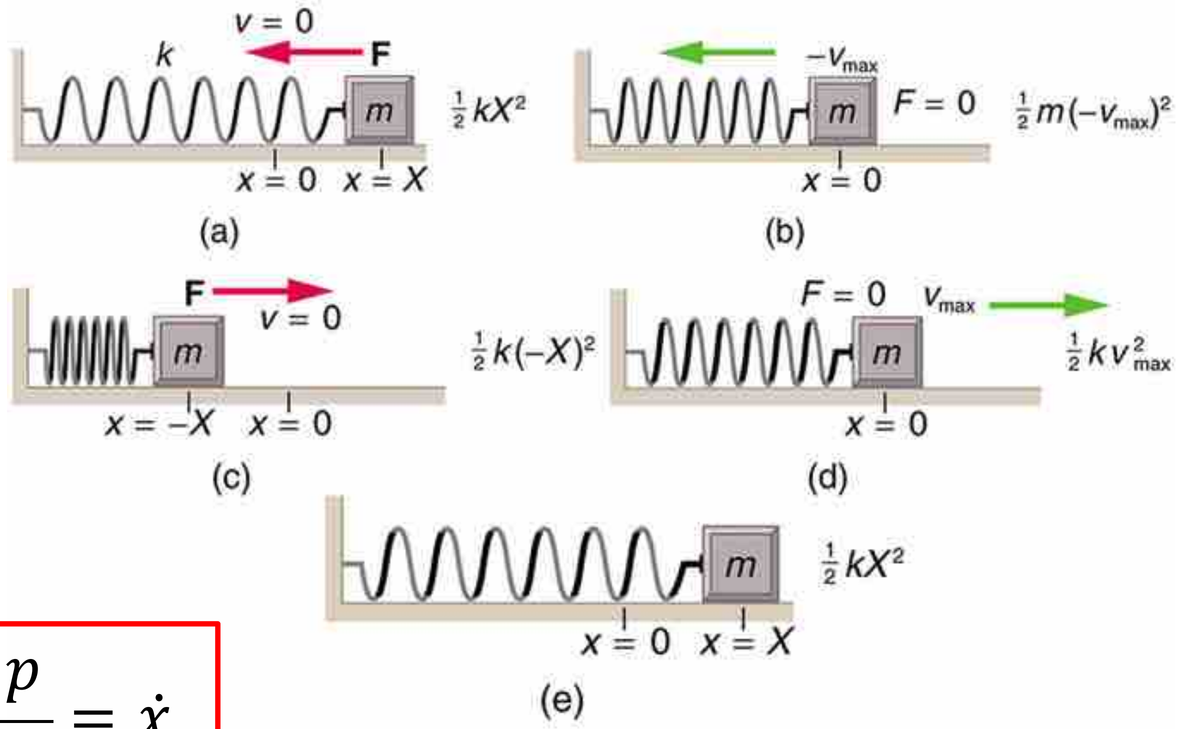
$$H = T + U$$

$$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

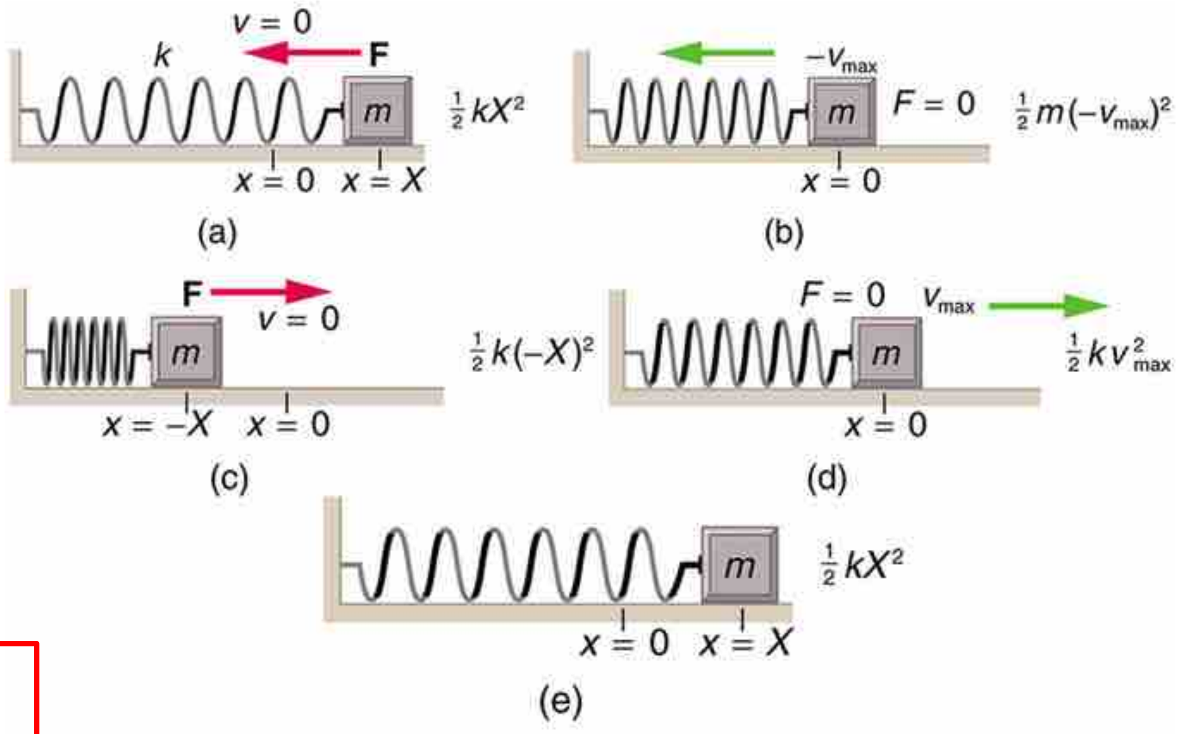
$$\frac{\partial H}{\partial x} = kx$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$



$$\frac{\partial H}{\partial x} = kx$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$



$$\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}$$

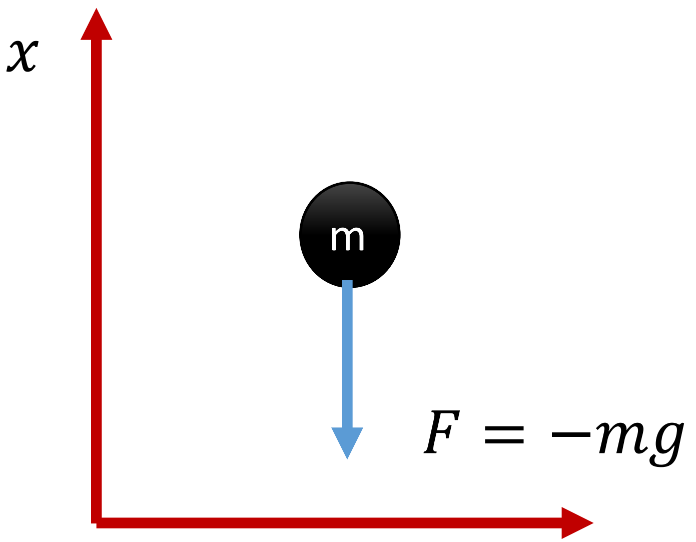
$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial x} = 0$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

$$\text{As } \omega^2 = \frac{k}{m}$$

Example 7.7: Find the equations of motion for a system of free fall particle using the Hamiltonian method.



$$H = T + U \;\Rightarrow\; H = \frac{1}{2}mv^2 + mgx$$

$$H = \frac{1}{2}m\dot{x}^2 + mgx \;\Rightarrow\; p = mv = m\dot{x}$$

$$p^2 = m^2\dot{x}^2 \;\Rightarrow\; H = \frac{p^2}{2m} + mgx$$

$$\frac{\partial H}{\partial x} = mg$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}$$

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial x} = 0$$

$$m\ddot{x} + mg = 0$$

OR

$$\ddot{x} = -g$$

$Equation\ of\ motion$

In general coordinate

Now for a system expressed in the generalized coordinates q_i and $\dot{q_i}$

q is a general coordinate $(x, y, z, \theta, , , \ldots)$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j$$

Canonical Equations

$$\frac{\partial H}{\partial p} = \dot{x}$$

$$\frac{\partial H}{\partial x} = -\dot{p}$$

X-direction

Example 7.8: Obtain Hamilton's equations of motion for one-dimensional harmonic oscillator and use them to find the differential equation.

$$T = \frac{P^2}{2m} \qquad U = \frac{1}{2}kx^2$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \frac{\partial H}{\partial q_j} = -\dot{p}_j$$

Canonical Equations

$$H = T + U$$

One-dimension

$$H = \frac{P^2}{2m} + \frac{1}{2}kx^2$$

$$\frac{\partial H}{\partial x} = -\dot{p} \qquad \frac{\partial H}{\partial p} = \dot{x}$$

Left-hand side

Left-hand side

$$\frac{\partial H}{\partial x} = kx \qquad \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$kx = -\dot{p}$$

$$\frac{p}{m} = \dot{x}$$

$$kx = -\frac{\partial p}{\partial t}$$

$$p = m\dot{x}$$

$$kx = -\frac{\partial (p)}{\partial t}$$

$$kx = -\frac{\partial (m\dot{x})}{\partial t}$$

$$kx = -m\ddot{x}$$

$$kx + m\ddot{x} = 0$$

Lagrangian VS Hamiltonian

$$L = T - V$$

$$L(q, \dot{q})$$ $$OR\ L(x, \dot{x})$$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{p}$$

$$H = T + V$$

$$H(q, p)$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$ $$\frac{\partial H}{\partial q_j} = -\dot{p}_j$$

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial q} = 0$$

Equation of Motion