CALCULUS I

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1. Sets:

~ Natural Numbers N = {1,2,3,···}

~ Integers Numbers Z = {··· ,−3,−2,−1,0,1,2,3,···} = $Z^- \cup \{0\} \cup Z^+$

~ Rational Numbers $\mathbb{Q} = \{\frac{a}{b} : a, b$ are integers numbers and b 6= 0}

~ Irrational Numbers I: Such as $\sqrt{2}$ and π are numbers which are not rational.

~ Real Numbers R: The set of rational and irrational numbers (R = Q ∪ I).

~ Complex Numbers C = {$x + yi$: x,y are real numbers and $i = \sqrt{-1}$} Clearly,

N ⊆ Z ⊆ Q ⊆ R ⊆ C

2. Operations With Real Numbers:

If a,b and c are real numbers, then:

1) $a + b \in$ R and $a \times b \in$ R (Closure law)

2) $a + b = b + a$ (Commutative law of addition)

3) $a \times b = b \times a$ (Commutative law of multiplication)

4) $a + (b + c) = (a + b) + c$ (associative law of addition)

5) $a \times (b \times c) = (a \times b) \times c$ (associative law of multiplication)

6) $a \times (b + c) = a \times b + a \times c$ (distributive law)

7) $a + 0 = 0 + a = a$ (0 is called the identity with respect to addition) $a \times 1 = 1 \times a = a$

(1 is called the identity with respect to multiplication)

8) For any a there is a number $x \in$ R such that $x + a = a + x = 0$, x is called the inverse of a with respect to addition and is denoted by $-a$.

9) For any a 6= 0 there is a number $x \in$ R such that $x \times a = a \times x = 1$, x is called the inverse of a with respect to multiplication and is denoted by a^{-1} or $\frac{1}{a}$.

3. Types of Intervals:

Interval Notation	Set definition	Name	Region on the Real Number Line
(a,b)	$\{x : a < x < b\}$	Open	
$[a,b]$	$\{x : a \le x \le b\}$	Closed	
$[a,b)$	$\{x : a \le x < b\}$	Half Open	
$(a,b]$	$\{x : a < x \le b\}$	Half Open	
(a,∞)	$\{x : x > a\}$	Open	
$[a,\infty)$	$\{x : x \ge a\}$	Closed	
$(-\infty,b)$	$\{x : x < b\}$	Open	
$(-\infty,b]$	$\{x : x \le b\}$	Closed	
$(-\infty,\infty)$	R	Open and Closed	

4. Inequalities:

If $a - b$ is a nonnegative number, we say that a is greater than or equal to b or b is less than or equal to a, and write, respectively $a \geq b$ or $b \leq a$. If there is no possibility that $a = b$, we write $a > b$ or $b < a$.

Theorem (4.1):

If a,b,c and d are any real numbers, then:

1) If $a < b$ and $b < c$, then $a < c$
   e.g., $4 < 5$ and $5 < 7 \Rightarrow 4 < 7$

2) If $a < b$, then $a \pm c < b \pm c$

   e.g., $10 < 13 \Rightarrow 10 + 3 < 13 + 3$ and $10 - 3 < 13 - 3$

3) If $a < b$, then $\left.\begin{array}{l} a \times c < b \times c \\ \dfrac{a}{c} < \dfrac{b}{c} \end{array}\right\}$ when $c > 0$

   e.g., $10 < 20 \Rightarrow 10 \times 3 < 20 \times 3 \Rightarrow 30 < 60$
   $\Rightarrow \dfrac{10}{5} < \dfrac{20}{5} \Rightarrow 2 < 4$

4) If $a < b$, then $\left.\begin{array}{l} a \times c > b \times c \\ \dfrac{a}{c} > \dfrac{b}{c} \end{array}\right\}$ when $c < 0$

   e.g., $10 < 20 \Rightarrow 10 \times -2 > 20 \times -2 \Rightarrow -20 > -40$
   $\Rightarrow \dfrac{10}{-2} > \dfrac{20}{-2} \Rightarrow -5 > -10$

5) If $a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$

   e.g., $3 < 5 \Rightarrow \dfrac{1}{3} > \dfrac{1}{5}$

6) If $a < b$ and $c < d$, then $a + c < b + d$ e.g., $3 < 5$ and $6 < 9 \Rightarrow 3 + 6 < 5 + 9$

Example (4.1): Find the solution set of the following inequalities.

1) $3 + 2x < 7$

Solution:

$$\Rightarrow 3 + 2x - 3 < 7 - 3 \Rightarrow 2x < 4 \Rightarrow \frac{2x}{2} < \frac{4}{2} \Rightarrow x < 2$$

∴ The solution = $\{x : x < 2\} = (-\infty, 2)$

2) $2 - 3x < 4 + 2x$

Solution:

$2 - 3x + 3x < 4 + 2x + 3x$ (adding to both sides $+3x$)

$\Rightarrow 2 < 4 + 5x \Rightarrow 2 - 4 < 4 + 5x - 4$ (adding to both sides -4)

$$-2 < 5x \Rightarrow \frac{-2}{5} < \frac{5x}{5}$$

$$\Rightarrow \frac{-2}{5} < x$$ (dividing both sides by 5)

∴ The solution = $\{x : x > \frac{-2}{5}\} = (\frac{-2}{5}, \infty)$

3) $2 < 3x - 1 \leq 11$

Solution:

$$\Rightarrow 2 + 1 < 3x - 1 + 1 \leq 11 + 1$$

$$\Rightarrow 3 < 3x \leq 12 \Rightarrow \frac{3}{3} < \frac{3x}{3} \leq \frac{12}{3} \Rightarrow 1 < x \leq 4$$

∴ The solution = $\{x : 1 < x \leq 4\} = (1, 4]$

4) $\frac{2}{x} < \frac{1}{4}$, $x \neq 0$ **Solution:**

x may be positive or negative.

Case 1: If $x > 0$

$\Rightarrow \frac{2}{\cancel{x}} \times \cancel{x} < \frac{1}{4} \times x \Rightarrow 2 < \frac{x}{4} \Rightarrow 2 \times 4 < \frac{x}{\cancel{4}} \times \cancel{4} \Rightarrow 8 < x$

$\therefore$ The solution = $\{x : x > 8\} = (8,\infty)$

Case 2: If $x < 0$

$\Rightarrow \frac{2}{\cancel{x}} \times \cancel{x} > \frac{1}{4} \times x \Rightarrow 2 > \frac{x}{4} \Rightarrow 2 \times 4 > \frac{x}{\cancel{4}} \times \cancel{4} \Rightarrow 8 > x$

$\therefore$ The solution = $\{x : x < 0\} = (-\infty,0)$

$\therefore$ The general solution is $(-\infty,0) \cup (8,\infty)$

5) $\frac{x-7}{x+3} > 2 \;, x \neq -3$

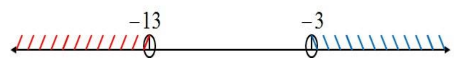
Solution:

Case 1: If $x + 3 > 0 \Rightarrow x > -3$ $x - 7$ XX+ 3)XX $> 2(x + 3) \Rightarrow x - 7 > 2x + 6 \Rightarrow x - 2x$

$> 6 + 7 \Rightarrow -x > 13$

$\Rightarrow$ XX+ 3XX(X x

$\Rightarrow x < -13$ this is false.



Case 2: If $x + 3 < 0 \Rightarrow x < -3$ $x - 7$ XX+ 3)XX $< 2(x + 3) \Rightarrow x - 7 < 2x + 6 \Rightarrow x - 2x$

$< 6 + 7 \Rightarrow -x < 13$

$\Rightarrow$ XX+ 3XX(X x

$\Rightarrow x > -13$.

$\therefore$ The solution is = $\{x : -13 < x < -3\} = (-13,-3)$

$\therefore$ The general solution is = $\{x : -13 < x < -3\} = (-13,-3)$

Exercises (4): Solve the following inequalities:

1) $\dfrac{x+4}{x-3} < 2$ 2) $\dfrac{-x}{x+5} < 1$ 3) $x^2 - 6x + 5 > 0$

4) $(x-1)^2(x+4) < 0$ 5) $5x - 2x^2 > 0$

5. Absolute Value:

Definition (5.1): If x and y any real numbers, then:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties:

1) $|-x| = |x|$

$\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, \quad y \neq 0$

2) $|xy| = |x||y|$

$|x| = \sqrt{x^2}$

3)

4)

5) $|x + y| \leq |x| + |y|$

6) $|x - y| \geq |x| - |y|$

7) $-|a| \leq a \leq |a|$

8) If $|x| \leq a$, then $-a \leq x \leq a$

9) If $|x| \geq a$, then $x \leq -a$ <u>or</u> $x \geq a$

Example (5.1):

1) $|4-8| = |-4| = 4$

2) $|4| + |-3| = 4 + 3 = 7$

3) $|4-8| = |4+(-8)| \leq |4| + |-8|$

4) $|4+8| = |4-(-8)| \geq |4| - |-8|$

Example (5.2): Solve $\left|x+\frac{1}{x}\right| > 2 \ , x \neq 0$

Solution:

$\Rightarrow \left|\frac{x^2+1}{x}\right| > 2 \Rightarrow \frac{|x^2+1|}{|x|} > 2$ (Since $x^2 + 1 > 0$)

$$\Rightarrow \frac{x^2+1}{|x|} > 2 \Rightarrow x^2+1 > 2|x| \Rightarrow x^2 - 2|x| + 1 > 0$$

$\Rightarrow |x|^2 - 2|x| + 1 > 0$ (Since $x^2 = |x|^2$)

$\Rightarrow (|x|-1)^2 > 0 \ , |x| \ 6= 1$

∴ The solution is the set of real number except $x = 1$, $x = -1$ and $x = 0$ ∴ The solution is $= (-\infty,-1) \cup (-1,0) \cup (0,1) \cup (1,\infty)$

Example (5.3): Solve $|x+3| \leq 5$

Solution:

$|x+3| \leq 5$ if and only if $-5 \leq x + 3 \leq 5$

$\Rightarrow -5 - 3 \leq x^{H} + 3_{H}^{H} -_{H} 3 \leq 5 - 3 \Rightarrow -8 \leq x \leq 2$

∴ The solution is $= \{x : -8 \leq x \leq 2\} = [-8,2]$

Exercises (5): Solve the following inequalities:

1) $|2x - 3| < |x + 2|$

2) $|2x + 1| > 2$

3) $|5 - 3x| < 2$

6. Functions:

Definition (6.1): A relation $f : X \longrightarrow Y$ is called function if and only if for each element $x \in X$, there exist a unique element $y \in Y$ such that $y = f(x)$.

~ The variable x in a function $y = f(x)$ is called the independent variable of the function f. The variable y whose value dependent on x, is called dependent variable of the function f.

~ If $y = f(x)$, then the set of all possible inputs ($x - values$) is called the domain of f and denoted by D_f or $Dom(f)$.

And the set of outputs ($y-values$) that result when x varies over the domain is called the range of f and denoted by R_f or $Ran(f)$.

Example (6.1): Find the domain and range of the following functions:

1) $f(x) = x - 2$

2) $f(x) = x^2 - 4$

4) $f(x) = |x|$

5) $f(x) = \dfrac{x^2 - 4}{x + 2}$

Solution:

1) $D_f = \mathbb{R}$ and $R_f = \mathbb{R}$

2) $D_f = \mathbb{R}$

Let $y = x^2 - 4$

$\Rightarrow x^2 = y + 4$

$\Rightarrow x = \mp\sqrt{y+4}$

If $y + 4 \geq 0 \Rightarrow y \geq -4$

$\therefore R_f = [-4, \infty)$

3) $f(x) = \sqrt{x-2}$

6) $f(x) = \dfrac{1}{(x-2)(x-3)}$

3) $x - 2 \geq 0 \Rightarrow x \geq 2$

$\therefore D_f = [2, \infty)$ and $R_f = [0, \infty)$

4) $D_f = \mathbb{R}$ and $R_f = [0, \infty)$

5) $x + 2 = 0 \Rightarrow x = -2$

$\therefore D_f = (-\infty, -2) \cup (-2, \infty)$

Since $f(x) = \dfrac{x^2 - 4}{x+2} = \dfrac{(x-2)(x+2)}{x+2} = x - 2$ for $x \neq -2$

$\Rightarrow f(x) = x - 2$

$\Rightarrow y = x - 2 \Rightarrow y = -2 - 2 = -4$

$\therefore R_f = (-\infty, -4) \cup (-4, \infty)$

6) $D_f = (-\infty, 2) \cup (2, 3) \cup (3, \infty)$

Let

$$y = \frac{1}{(x-2)(x-3)} \Rightarrow y = \frac{1}{x^2 - 5x + 6} \Rightarrow y(x^2 - 5x + 6) = 1$$

$$\Rightarrow x^2 - 5x + 6 = \frac{1}{y} \Rightarrow x^2 - 5x + \left(6 - \frac{1}{y}\right) = 0$$

$$\Rightarrow x = \frac{5 \pm \sqrt{25 - 4\left(6 - \frac{1}{y}\right)}}{2} = \frac{5 \pm \sqrt{25 - 24 + \frac{4}{y}}}{2} = \frac{5 \pm \sqrt{1 + \frac{4}{y}}}{2}$$

If $1 + \frac{4}{y} \geq 0 \Rightarrow \frac{4}{y} \geq -1$

Case 1: If $y > 0$

$\Rightarrow 4 \geq -y \Rightarrow y \geq -4 \Rightarrow (0,\infty)$

Case 2: If $y < 0$

$\Rightarrow 4 \leq -y \Rightarrow y \leq -4 \Rightarrow (-\infty,-4]$

$\therefore R_f = (-\infty,-4] \cup (0,\infty)$

Example (6.2): Find the domain of the following functions:

1) $f(x) = \dfrac{3x}{x^2 - 4x - 12}$ 2) $f(x) = \dfrac{\sqrt{x-1}}{x^2 + 4}$ 3) $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$

Solution:

1) $x^2 - 4x - 12 = 0 \Rightarrow (x - 6)(x + 2) = 0 \Rightarrow x = 6, x = -2$

$\therefore D_f = (-\infty,-2) \cup (-2,6) \cup (6,\infty)$

2) $x - 1 \geq 0 \Rightarrow x \geq 1$

$\therefore D_f = [1,\infty)$

3) $x^2 - 4 > 0 \Rightarrow x^2 > 4$ this is true if $x < -2$ or $x > 2$

$\therefore D_f = (-\infty,-2) \cup (2,\infty)$

Exercises (6.1): Find D_f and R_f of the following functions:

1) $f(x) = -x^2 + 4$ 2) $f(x) = \sqrt{x}$ 3) $f(x) = \sin(x)$

4) $f(x) = \cos^2(x)$

Exercises (6.2): Find D_f of the following functions:

1) $f(x) = \begin{cases} 3 - x & \text{if } x \le 1 \\ 5x - 3 & \text{if } x > 1 \end{cases}$ 2) $f(x) = \dfrac{|x|}{x}$

Definition (6.2): Let $f(x)$ be a function with domain D_f and $g(x)$ be a function with domain D_g and define: $D = D_f \cap D_g$, then:

1) $(f + g)(x) = f(x) + g(x)$ with domain D
2) $(f - g)(x) = f(x) - g(x)$ with domain D
3) $(f.g)(x) = f(x).g(x)$ with domain D
4) $(f/g)(x) = f(x)/g(x)$ with domain D and $g(x) \neq 0$

Example (6.3): Let $f(x) = 1 + \sqrt{x - 2}$ and $g(x) = x - 3$, find $(f + g)(x)$, $(f - g)(x)$, $(f.g)(x)$, $(f/g)(x)$ and state the domain of $f + g$, $f - g$, $f.g$, f/g.

Solution:

1) $(f + g)(x) = f(x) + g(x) = 1 + \sqrt{x - 2} + x - 3 = x - 2 + \sqrt{x - 2}$
2) $(f - g)(x) = f(x) - g(x) = 1 + \sqrt{x - 2} - x + 3 = 4 - x + \sqrt{x - 2}$

3) $(f.g)(x) = f(x).g(x) = (1 + \sqrt{x-2})(x-3) = x - 3 + (x-3)\sqrt{x-2}$

$$(f/g)(x) = f(x)/g(x) = \frac{1+\sqrt{x-2}}{x-3}$$

$$\because f(x) = 1 + \sqrt{x-2} \Rightarrow x - 2 \geq 0 \Rightarrow x \geq 2$$

$$\therefore D_f = [2, \infty)$$

4) $\because g(x) = x - 3$

$\therefore D_g = (-\infty, \infty)$

$\therefore D = D_f \cap D_g = [2,\infty) \cap (-\infty,\infty) = [2,\infty)$

$\therefore Dom(f+g, f-g, f.g) = D = [2,\infty)$

$Dom(f/g) = [2,3) \cup (3,\infty)$

Exercises (6.3): Let $f(x) = 2\sqrt{x-1}$ and $g(x) = \sqrt{x-1}$ find the domain of $f+g, f-g, f.g,$ and f/g.

7. Composition of Function:

Definition (7.1): The composition function $(f \circ g)$ defined by $(f \circ g)(x) = f(g(x))$ the notation $(f \circ g)$ is read (f follows g or the composition of f and g).

$f : X \longrightarrow Y,\ g : Y \longrightarrow Z \Rightarrow f \circ g : X \longrightarrow Z$

Example (7.1): Let $f(x) = 2x + 1$ and $g(x) = x^2 - x$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1) $(f \circ g)(x) = f(g(x)) = f\left(x^2 - x\right) = 2(x^2 - x) + 1 = 2x^2 - 2x + 1$

2) $(g \circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^2 - (2x+1)$

Example (7.2): Let $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{x^2+3}$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1) $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x^2+3}\right) = \sqrt{\sqrt{x^2+3}-3}$$

2) $$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x-3}\right) = \sqrt{(\sqrt{x-3})^2+3} = \sqrt{x-3+3} = \sqrt{x}$$

Exercises (7): Find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the following:

1) $f(x) = x^2$, $g(x) = \sqrt{1-x}$

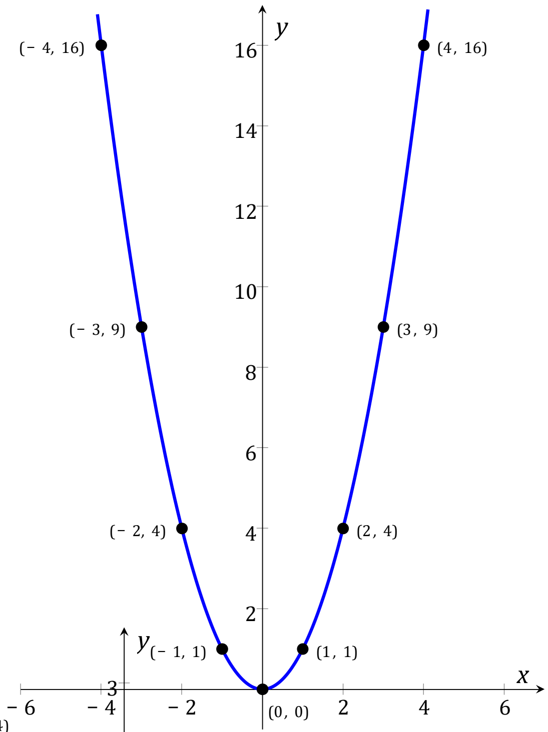
2) $$f(x) = \frac{1+x}{1-x} \text{ , } g(x) = \frac{x}{1-x}$$

3) $$f(x) = \frac{x}{1+x^2} \text{ , } g(x) = \frac{1}{x}$$

8. Graph of a Function:

A function f establishes a set of ordered pairs (x,y) of real number. The plot of these pairs $(x,f(x))$ in a coordinate system is the graph of f.

Example (8.1): Sketch a graph of the function $f(x) = x^2$



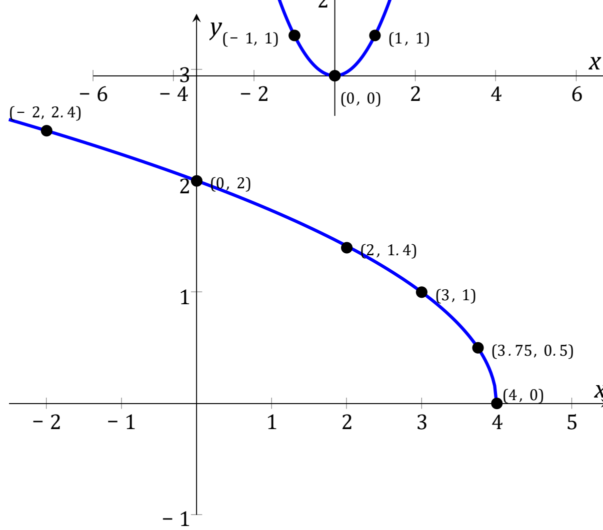
Solution:

$D_f = R$

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Make a table values of x from the domain.

Example (8.2): Sketch a graph of the function $f(x) = \sqrt{4 - x}$



Solution:

$4 - x \geq 0 \Rightarrow 4 \geq x \Rightarrow D_f = (-\infty, 4]$

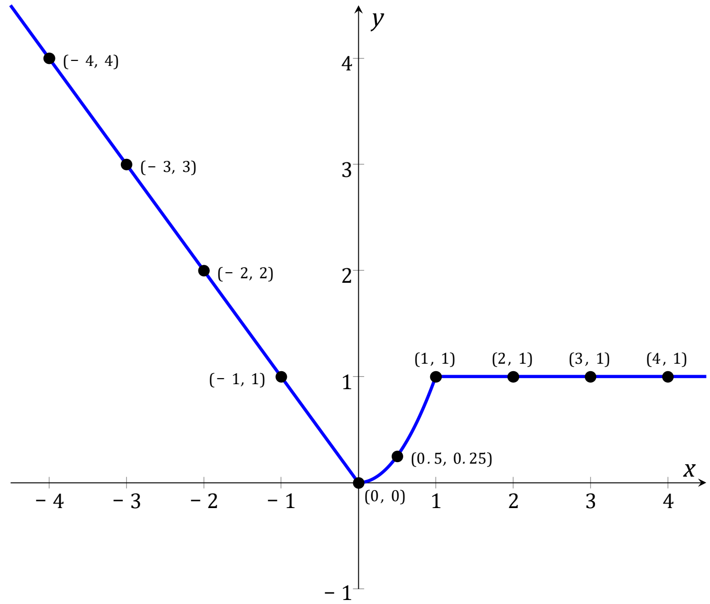
x	4	3.75	3	2	0	-2
y	0	0.5	1	1.4	2	2.4

Make a table values of x from the domain.

Example (8.3): Sketch a graph of the function

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ [illegible]\ 1 & \text{if } x > 1 \end{cases}$$

$\{ -x$ if $x < 0$



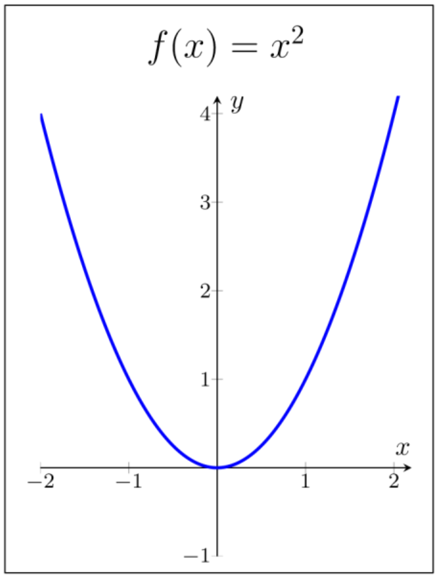
Solution:

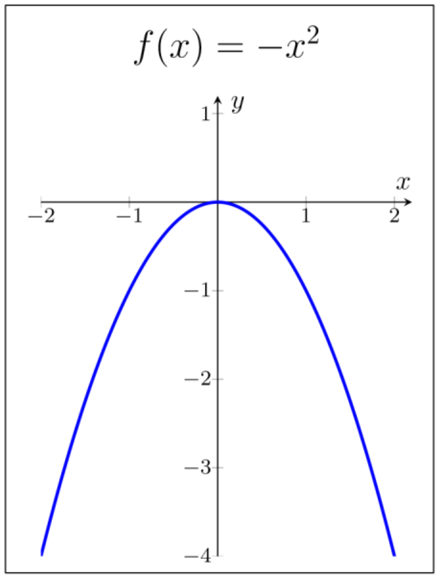
D_f = R

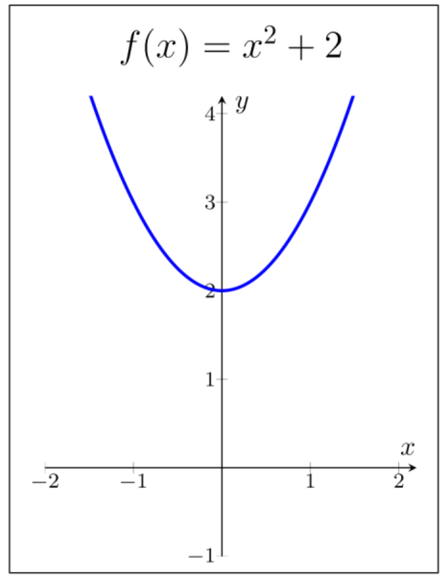
x	-4	-3	-2	-1	0	0.5	1	2	3	4
y	4	3	2	1	0	0.25	1	1	1	1

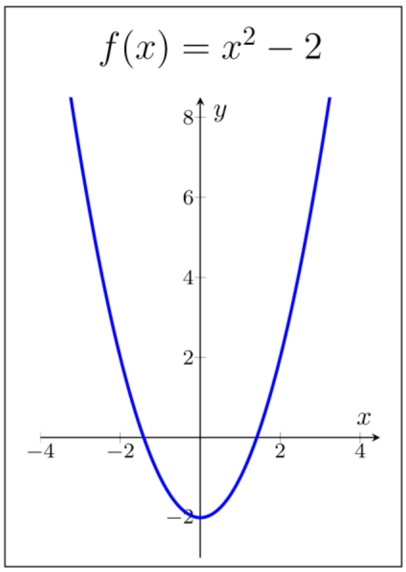
Make a table values of *x* from the domain.

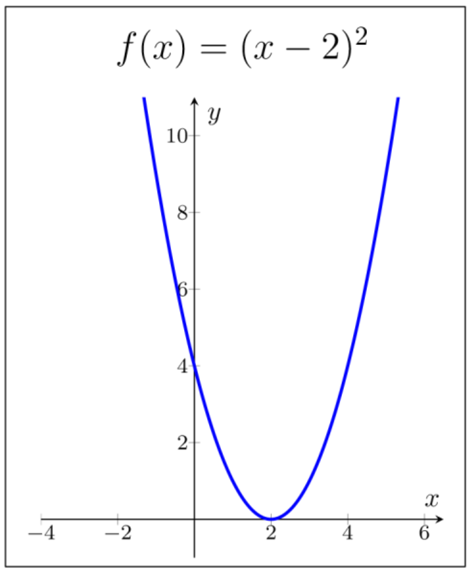
Remark (8.1):

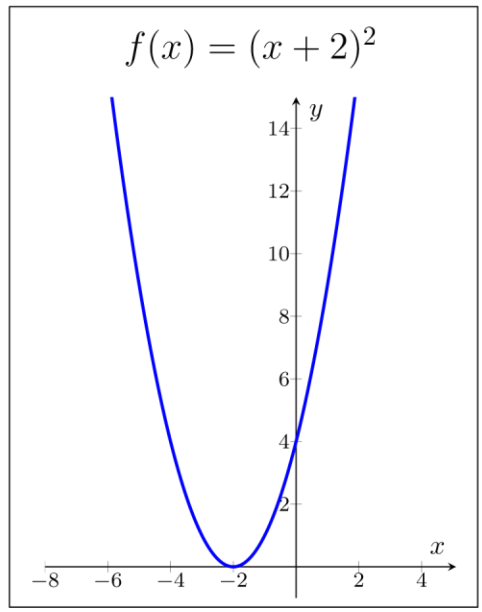












9. Even Functions and Odd Functions:

Definition (9.1): A function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ for every x in the function's domain. It is odd function of x if $f(-x) = -f(x)$ for every x in the function's domain.

Example (9.1): $f(x) = x^2$ is even function since $f(-x) = (-x)^2 = x^2 = f(x)$

$f(x) = x^3$ is odd function since $f(-x) = (-x)^3 = -x^3 = -f(x)$

10. Test of Symmetric:

To test for various kinds of symmetry we state the following rules:

i. about $x - axis$ replace y by $-y$ $(-y \dashrightarrow y)$ in its equation produces an equivalent equation.

ii. about $y - axis$ replace x by $-x$ $(-x \dashrightarrow x)$ in its equation produces an equivalent equation.

iii. about the origin point

replace x by $-x$ and y by $-y$ $(-x \dashrightarrow x \wedge -y \dashrightarrow y)$ in its equation produces an equivalent equation.

Definition (10.1): A line $y = b$ is a horizontal asymptote of the graph of the relation if the distance between the curve and the line $y = b$ tends to zero as the curve continuous upwards beyond all bound.

Definition (10.2): A line $x = a$ is a vertical asymptote of the graph of the relation if the distance between the curve and the line $x = a$ tends to zero as the curve continuous upwards beyond all bound.

~ To test a horizontal asymptote, we flow the following:

1) We solve x in terms of y.

2) If x is given of form $x = \frac{r(y)}{t(y)}$ and find all those values of y for which $t(y) = 0$ and $r(y)$ 6= 0 then the values of y which satisfy $t(y) = 0$ are horizontal asymptotes of the graph.

~ To test a vertical asymptote, we flow the following:

1) We solve y in terms of x.

2) If y is given of form $y = \frac{g(x)}{h(x)}$ and find all those values of x for which $h(x) = 0$ and $g(x)$ 6= 0 then the values of x which satisfy $h(x) = 0$ are vertical asymptotes of the graph.

Example (10.1): Sketch a graph of the following functions:

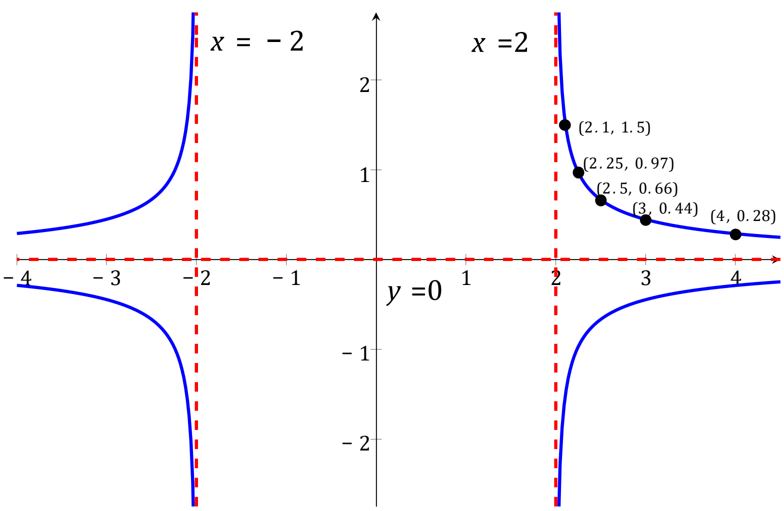
1) $(x^2 - 4)y^2 = 1$ 2) $x^2y = x - 3$ (H.W)

Solution 1: *Dom* = $(-\infty,-2) \cup (2,\infty)$ **Test of Symmetric:**

i. about $x - axis$ $(-y \longrightarrow y) \Rightarrow (x^2 - 4)(-y)^2 = 1 \Rightarrow (x^2 - 4)y^2 = 1$ ∴ Symmetric about $x - axis$.

ii. about $y - axis$ $(-x \longrightarrow x) \Rightarrow ((-x)^2 - 4)y^2 = 1 \Rightarrow (x^2 - 4)y^2 = 1$ ∴ Symmetric about $y - axis$.

iii. From (i) and (ii) we get symmetric about the origin point.



Test of Asymptotes:

1) $$(x^2 - 4)y^2 = 1 \Rightarrow x^2y^2 - 4y^2 = 1 \Rightarrow x^2y^2 = 1 + 4y^2 \Rightarrow x = \pm\frac{\sqrt{1 + 4y^2}}{y}$$

$\Rightarrow y = 0$ is a horizontal asymptote.

2) $$(x^2 - 4)y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{x^2 - 4}} \Rightarrow \text{If } \sqrt{x^2 - 4} = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$$

∴ $x = 2$ and $x = -2$ are vertical asymptotes.

x	-4	-3	-2.5	-2.25	-2.1	2.1	2.25	2.5	3	4
y	±0.28	±0.44	±0.66	±0.97	±1.5	±1.5	±0.97	±0.66	±0.44	±0.28

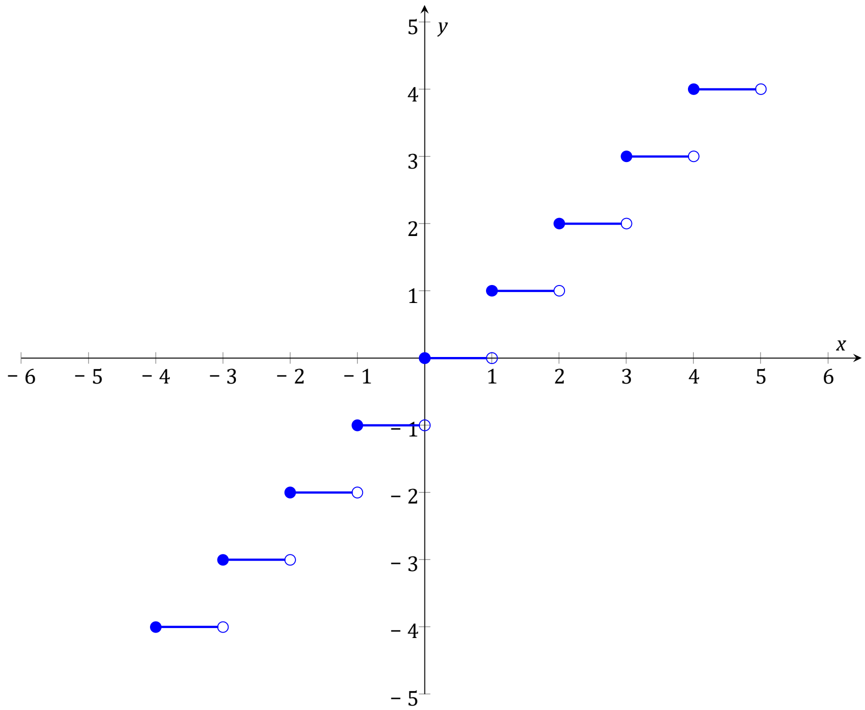
11. Greatest Integer Function:

$$f(x) = [x] = \begin{cases} \vdots & \vdots \\ -2 & \text{if } -2 \le x < -1 \\ -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ \vdots & \vdots \end{cases}$$

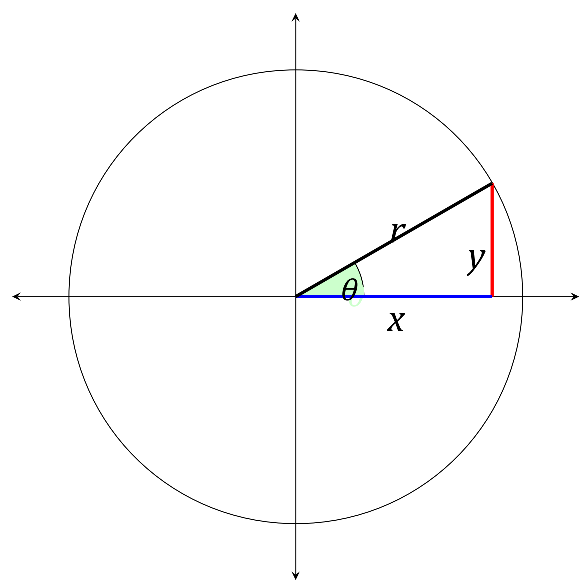
$D_f = \mathrm{R}$ and $R_f = \mathrm{Z}$

For Example:

$f(0.5) = [0.5] = 0$ $f(1.9) = [1.9] = 1$ $f(2.4) = [2.4] = 2$ $f(-1.2) = [-1.2] = -2$



12. Trigonometric Functions:



1) $\sin(\theta) = \frac{y}{r}$

2) $\cos(\theta) = \frac{x}{r}$

3) $\tan(\theta) = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin(\theta)}{\cos(\theta)}$

4) $\cot(\theta) = \frac{x}{y} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{\cos(\theta)}{\sin(\theta)}$

5) $\sec(\theta) = \frac{r}{x} = \frac{1}{\cos(\theta)}$

6) $\csc(\theta) = \frac{r}{y} = \frac{1}{\sin(\theta)}$

7) $\because x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \Rightarrow \cos^2(\theta) + \sin^2(\theta) = 1$

8) $\because x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \Rightarrow$ $\cot^2(\theta) + 1 = \csc^2(\theta)$

9) $\because x^2+y^2=r^2 \Rightarrow 1+\frac{y^2}{x^2}=\frac{r^2}{x^2} \Rightarrow 1+\tan^2(\theta)=\sec^2(\theta)$

Definition (12.1): A function $f(x)$ is periodic with period $\rho > 0$ if $f(x+\rho) = f(x)$ for every value of x.

Example (12.1): $f(x) = \sin(x)$, $f(x) = \cos(x)$ are periodic function such that $\rho = 2\pi$ *i.e* :

$\sin(\theta) = \sin(\theta + 2\pi)$

$\cos(\theta) = \cos(\theta + 2\pi)$

In general:

$\sin(\theta) = \sin(\theta + 2n\pi)$, $n = \mp1, \mp2, \mp3, \cdots$ $\cos(\theta) =$

$\cos(\theta + 2n\pi)$, $n = \mp1, \mp2, \mp3, \cdots$ **Remark (12.1):**

1) $\sin(-\theta) = -\sin(\theta)$ odd function.
2) $\cos(-\theta) = \cos(\theta)$ even function.
3) $\tan(-\theta) = -\tan(\theta)$ odd function.
4) $\cot(-\theta) = -\cot(\theta)$ odd function.
5) $\sec(-\theta) = \sec(\theta)$ even function.
6) $\csc(-\theta) = -\csc(\theta)$ odd function.

Properties of Trigonometric Functions:

$$\sin(\theta+\frac{\pi}{2}) = \cos(\theta)$$

$$\cos(\theta+\frac{\pi}{2}) = -\sin(\theta)$$

1)

2)

3) $\sin(x \mp y) = \sin(x)\cos(y) \mp \sin(y)\cos(x)$

4) $\cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$

5) $\sin(2x) = 2\sin(x)\cos(x)$

6) $\cos(2x) = \cos^2(x) - \sin^2(x)$

7) $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

8) $\tan(x \mp y) = \dfrac{\tan(x) \mp \tan(y)}{1 \pm \tan(x)\tan(y)}$

9) $\sin(x)\sin(y) = \dfrac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

10) $\cos(x)\cos(y) = \dfrac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

11) $\sin(x)\cos(y) = \dfrac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$

Example (12.2): Prove that $\dfrac{\cos(\theta)}{\sin(\theta)\cot(\theta)} = 1$

Proof:

$$\frac{\cos(\theta)}{\cancel{\sin(\theta)}\frac{\cos(\theta)}{\cancel{\sin(\theta)}}} = \frac{\cos(\theta)}{\cancel{\cos(\theta)}} = 1$$

Example (12.3): Prove that $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{1+\sin(\theta)}{\cos(\theta)}$

Proof:

$$\frac{\cos(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)} = \frac{\cos(\theta)(1+\sin(\theta))}{1-\sin^2(\theta)} = \frac{\cancel{\cos(\theta)}(1+\sin(\theta))}{\cancel{\cos^2(\theta)}^{\cos(\theta)}} = \frac{1+\sin(\theta)}{\cos(\theta)}$$

$2\cot(x)$ **Example**

(12.4): Solve 1 +__________ $\cot_2(x)$

Solution:

$$\frac{2\cot(x)}{1+\cot^2(x)} = \frac{2\cot(x)}{\csc^2(x)} = \frac{\frac{2\cos(x)}{\sin(x)}}{\frac{1}{\sin^2(x)}} = 2\cos(x)\sin(x) = \sin(2x)$$

Exercises (12): Prove that

$$\frac{\tan^2(\theta)+1}{\sec(\theta)} = \sec(\theta)$$

$$\frac{\cos(\theta)+1}{\tan^2(\theta)} = \frac{\cos(\theta)}{\sec(\theta)-1}$$

1)

2)

$$\frac{\tan(\theta) - \cot(\theta)}{\sin(\theta)\cos(\theta)}$$ 3) $= \sec^2(\theta) - \csc^2(\theta)$

$$\frac{\sec^2(\theta)-1}{\sec^2(\theta)} = \sin^2(\theta)$$

$$\frac{\tan^2(\theta)\csc^2(\theta)-1}{\sec^2(\theta)} = \sin^2(\theta)$$

4)

5) **Definition (12.2):** If

the functions *f* and *g* satisfy the

two conditions:

i. $g(f(x)) = x$ for every x in the domain of f.

ii. $f(g(y)) = y$ for every y in the domain of g. then we call f an inverse function of g and g an inverse function for f.

13. Inverse of Trigonometric Functions:

1) If $y = \sin(x) \Rightarrow x = \sin^{-1}(y)$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-1 \leq y \leq 1$
2) If $y = \cos(x) \Rightarrow x = \cos^{-1}(y)$ where $0 \leq x \leq \pi$, $-1 \leq y \leq 1$
3) If $y = \tan(x) \Rightarrow x = \tan^{-1}(y)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\forall y \in \mathbb{R}$
4) If $y = \cot(x) \Rightarrow x = \cot^{-1}(y)$ where $0 < x < \pi$, $\forall y \in R$
5) If $y = \sec(x) \Rightarrow x = \sec^{-1}(y)$ where $0 \leq x < \frac{\pi}{2} \bigcup \frac{\pi}{2} < x \leq \pi$, $|y| \geq 1$
6) If $y = \csc(x) \Rightarrow x = \csc^{-1}(y)$ where $-\frac{\pi}{2} \leq x < 0 \bigcup 0 < x \leq \frac{\pi}{2}$, $|y| \geq 1$

Remark (13.1):

$$\sin^{-1}(x) \neq (\sin(x))^{-1} = \frac{1}{\sin(x)}$$

Example (13.1): $\sin(90) = 1 \Rightarrow \sin^{-1}(\sin(90)) = \sin^{-1}(1) \Rightarrow \sin^{-1}(1) = 90$

Example (13.2): Find the exact values of $\sin^{-1}(\frac{1}{\sqrt{2}})$.

Solution:

Let $y = \sin^{-1}(\frac{1}{\sqrt{2}}) \Rightarrow \sin(y) = \frac{1}{\sqrt{2}} \Rightarrow y = \frac{\pi}{4}$

Example (13.3): Find the exact values of $\sin^{-1}(\frac{1}{2})$.

Solution:

Let $y = \sin^{-1}(\frac{1}{2}) \Rightarrow \sin(y) = \frac{1}{2} \Rightarrow y = \frac{\pi}{6}$

Lemma (13.1):

$$\sec^{-1}(x) = \cos^{-1}(\tfrac{1}{x})$$

Solution:

Let $y = \sec^{-1}(x) \Rightarrow \sec(y) = x \Rightarrow \dfrac{1}{\cos(y)} = x \Rightarrow \cos(y) = \dfrac{1}{x} \Rightarrow y = \cos^{-1}(\tfrac{1}{x})$

$\Rightarrow \sec^{-1}(x) = \cos^{-1}(\tfrac{1}{x})$

Example (13.4): Prove that $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$

Proof:

$\sin^{-1}(x) = \dfrac{\pi}{2} - \cos^{-1}(x)$

Let $y = \dfrac{\pi}{2} - \cos^{-1}(x) \Rightarrow \cos^{-1}(x) = \dfrac{\pi}{2} - y \Rightarrow x = \cos\left(\dfrac{\pi}{2} - y\right) \Rightarrow x = \sin(y)$

$\Rightarrow y = \sin^{-1}(x)$

$\Rightarrow \dfrac{\pi}{2} - \cos^{-1}(x) = \sin^{-1}(x) \Rightarrow \sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$

14. Exponential Functions:

A function of the form $f(x) = b^x$, where $b > 0$ and b 6= 1, is called an exponential function with base b.

~ D_f = R and $R_f = (0,\infty)$

Example (14.1): $f(x) = 2^x \ , \ f(x) = \left(\dfrac{1}{2}\right)^x \ , \ f(x) = \pi^x$

Properties of Exponential Functions:

1) $a^x \times a^y = a^{x+y}$

2) $\dfrac{a^x}{a^y} = a^{x-y}$

3) $(a^x)^y = a^{xy}$

4) $(ab)^x = a^x b^x$

5) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

6) $a^{-x} = \frac{1}{a^x}$

7) $a^0 = 1$

8) $a^{\frac{1}{x}} = \sqrt[x]{a}$

9) $a^{\infty} = \infty$, $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0$

Remark (14.1):

The function $f(x) = e^x$ is called the natural exponential function, such that $e = 2.7$

15. Logarithmic Functions:

Is inverse of the exponential functions, $y = b^x$ is equivalent to $x = \log_b y$ if $y > 0$ and x is any real number.

~ b is called the base of the logarithmic.

~ If $b = 10 \Rightarrow x = \log y$ common logarithmic.

~ If $b = e \Rightarrow x = \log_e y = \ln(y)$ natural logarithmic.

~ Domain of logarithmic function is $(0,\infty)$ and it is range is R.

Properties of Logarithmic Functions:

If $b > 0$, b 6= 1 , $a > 0$, $c > 0$ and r is any real number, then

$$\log_b(ac) = \log_b a + \log_b c$$

$$\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$$

$$\log_b a^r = r\log_b a$$

1)

2)

3)

$$\log_b 1 = 0$$

$$\log_b\left(\frac{1}{c}\right) = -\log_b c$$

4)

5)

6) $\log_b x$ is undefine for $x < 0$

7) $\log_b b = 1$

8) $\ln(e^x) = x$ for every x

9) $e^{\ln(x)} = x$

10) $\log_b x = \dfrac{\ln(x)}{\ln(b)}$

11) $\log_b b^x = x$ for every x

Example (15.1): Find $\log \dfrac{xy^5}{\sqrt{z}}$ **Solution:**

$$\log \frac{xy^5}{\sqrt{z}} = \log(xy^5) - \log(\sqrt{z}) = \log x + \log y^5 - \log z^{\frac{1}{2}}$$
$$= \log x + 5\log y - \frac{1}{2}\log z$$

Example (15.2): Find $\frac{1}{3}\ln(x) - \ln(x^2-1) + 2\ln(x+3)$

Solution:

$$\frac{1}{3}\ln(x) - \ln(x^2-1) + 2\ln(x+3) = \ln(x)^{\frac{1}{3}} - \ln(x^2-1) + \ln(x+3)^2$$
$$= \ln\left(x^{\frac{1}{3}}(x+3)^2\right) - \ln(x^2-1)$$
$$= \ln\left(\frac{\sqrt[3]{x}(x+3)^2}{x^2-1}\right)$$

Example (15.3): Find *x* such that

1) $\log x = 2$ 5) $(x)^{\log(x)} = 100x$ (H.W)

2) $\ln(x+1) = 5$

3) $5^x = 7$

4) $\frac{e^x - e^{-x}}{2} = 1 \ (\mathcal{H}.\mathcal{W})$

Solution:

1) $\log x = 2 \Rightarrow \frac{\ln(x)}{\ln(10)} = 2 \Rightarrow \ln(x) = 2\ln(10) \Rightarrow \ln(x) = \ln(10)^2 \Rightarrow e^{\ln(x)} = e^{\ln(100)}$
$\Rightarrow x = 100$

2) $\ln(x+1) = 5 \Rightarrow e^{\ln(x+1)} = e^5 \Rightarrow x+1 = e^5 \Rightarrow x = e^5 - 1$

3) $5^x = 7 \Rightarrow \ln(5^x) = \ln(7) \Rightarrow x\ln(5) = \ln(7) \Rightarrow x = \frac{\ln(7)}{\ln(5)}$

16. Hyperbolic Functions:

1) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ where D_f= R , R_f= R

2) $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ where D_f= R , R_f= $[1,\infty)$

3) $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{\sinh(x)}{\cosh(x)}$ where D_f= R , R_f= $(-1,1)$

4) $\coth(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{1}{\tanh(x)}$ where D_f= R\{0} , R_f= R\(−1,1)

5) $\operatorname{sech}(x) = \dfrac{2}{e^x + e^{-x}} = \dfrac{1}{\cosh(x)}$ where D_f= R , R_f= $(0,1]$

6) $\operatorname{csch}(x) = \dfrac{2}{e^x - e^{-x}} = \dfrac{1}{\sinh(x)}$ where D_f= R\{0} , R_f= R\{0}

7) $\cosh^2(x) - \sinh^2(x) = 1$ **Proof:**

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$
$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1$$

8) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

9) $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

Remark (16.1):

1) $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2} = \dfrac{-(e^x - e^{-x})}{2} = -\sinh(x)$ odd function.

2) $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$ even function.

3) $\tanh(-x) = -\tanh(x)$ odd function.

4) $\coth(-x) = -\coth(x)$ odd function.

5) $\operatorname{sech}(-x) = \operatorname{sech}(x)$ even function.

6) $\operatorname{csch}(-x) = -\operatorname{csch}(x)$ odd function.

Properties of Hyperbolic Function:

1) $\sinh(x \mp y) = \sinh(x)\cosh(y) \mp \sinh(y)\cosh(x)$

$$\cosh(x \mp y) = \cosh(x)\cosh(y) \mp \sinh(x)\sinh(y)$$

$$\tanh(x \mp y) = \frac{\tanh(x) \mp \tanh(y)}{1 \mp \tanh(x)\tanh(y)}$$

2)

3)

4) $\sinh(2x) = 2\sinh(x)\cosh(x)$

5) $\cosh(2x) = \sinh^2(x) + \cosh^2(x)$ *or* $= 2\sinh^2(x) + 1$

or $= 2\cosh^2(x) - 1$

$$\sinh^2(x) = \frac{\cosh(2x) - 1}{2}$$

$$\cosh^2(x) = \frac{\cosh(2x) + 1}{2}$$

6)

7)

Example (16.1): Let $\cosh(x) = 5$, $x > 0$, find $\sinh(x)$, $\tanh(x)$, $\coth(x)$, $\operatorname{sech}(x)$ and $\operatorname{csch}(x)$

Solution:

$\because \cosh^2(x) - \sinh^2(x) = 1 \Rightarrow 25 - \sinh^2(x) = 1 \Rightarrow \sinh^2(x) = 25 - 1 \Rightarrow \sinh(x) = \sqrt{24}$

$$\therefore \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\sqrt{24}}{5} \quad , \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{5}{\sqrt{24}}$$

$$\therefore \operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{1}{5} \quad , \quad \operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{1}{\sqrt{24}}$$

Example (16.2): Prove that

1) $\cosh(x) + \sinh(x) = e^x$

2) $\cosh(x) - \sinh(x) = e^{-x}$ (H.W) **Proof:**

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

Example (16.3): Prove that $\tanh(\frac{1}{2}\ln(x)) = \frac{x-1}{x+1}$

Proof:

$$\tanh(\tfrac{1}{2}\ln(x)) = \frac{e^{\frac{1}{2}\ln(x)} - e^{-\frac{1}{2}\ln(x)}}{e^{\frac{1}{2}\ln(x)} + e^{-\frac{1}{2}\ln(x)}} = \frac{e^{\ln(x^{\frac{1}{2}})} - e^{\ln(x^{-\frac{1}{2}})}}{e^{\ln(x^{\frac{1}{2}})} + e^{\ln(x^{-\frac{1}{2}})}} = \frac{e^{\ln(\sqrt{x})} - e^{\ln(\frac{1}{\sqrt{x}})}}{e^{\ln(\sqrt{x})} + e^{\ln(\frac{1}{\sqrt{x}})}} = \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$= \frac{\frac{x-1}{\sqrt{x}}}{\frac{x+1}{\sqrt{x}}} = \frac{x-1}{x+1}$$

Example (16.4): Prove that

1) $\tanh(x+y) = \dfrac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$

2) $\tanh(2x) = \dfrac{2\tanh(x)}{1 + \tanh^2(x)}$ (H.W)

Proof:

$$\frac{\frac{e^x-e^{-x}}{e^x+e^{-x}}+\frac{e^y-e^{-y}}{e^y+e^{-y}}}{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}\frac{e^y-e^{-y}}{e^y+e^{-y}}} = \frac{\frac{(e^x-e^{-x})(e^y+e^{-y})+(e^y-e^{-y})(e^x+e^{-x})}{(e^x+e^{-x})(e^y+e^{-y})}}{\frac{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})}{(e^x+e^{-x})(e^y+e^{-y})}}$$

$$= \frac{e^{x+y}+e^{x-y}-e^{y-x}-e^{-(x+y)}+e^{x+y}+e^{y-x}-e^{x-y}-e^{-(x+y)}}{e^{x+y}+e^{x-y}+e^{y-x}+e^{-(x+y)}+e^{x+y}-e^{x-y}-e^{y-x}+e^{-(x+y)}}$$

$$= \frac{2e^{x+y}-2e^{-(x+y)}}{2e^{x+y}+2e^{-(x+y)}} = \frac{e^{x+y}-e^{-(x+y)}}{e^{x+y}+e^{-(x+y)}} = \tanh(x+y)$$

17. Inverse of Hyperbolic Functions:

1) If $y = \sinh(x) \Rightarrow x = \sinh^{-1}(y)$ where $D_f = \mathbb{R}$, $R_f = \mathbb{R}$

2) If $y = \cosh(x) \Rightarrow x = \cosh^{-1}(y)$ where $D_f = [1,\infty)$, $R_f = [0,\infty)$

3) If $y = \tanh(x) \Rightarrow x = \tanh^{-1}(y)$ where $D_f = (-1,1)$, $R_f = \mathbb{R}$

4) If $y = \coth(x) \Rightarrow x = \coth^{-1}(y)$ where $D_f = \mathbb{R}\backslash[-1,2]$, $R_f = \mathbb{R}\backslash\{0\}$

5) If $y = \text{sech}(x) \Rightarrow x = \text{sech}^{-1}(y)$ where $D_f = (0,1]$, $R_f = \mathbb{R}$

6) If $y = \text{csch}(x) \Rightarrow x = \text{csch}^{-1}(y)$ where $D_f = \mathbb{R}\backslash\{0\}$, $R_f = \mathbb{R}\backslash\{0\}$ **Relations Between Functions:**

$$\sinh^{-1}(x) = \ln(x+\sqrt{x^2+1})$$

1) $$\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$$

2)

3) $$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \ , \ |x|<1$$

4) $\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x| > 1$

5) $\operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \le 1$

6) $\operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$, $\forall x \in \mathbb{R}\backslash\{0\}$

Proof:

1) Let $y = \sinh^{-1}(x) \Rightarrow x = \sinh(y) \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - e^{-y}$

$e^y - 2x - e^{-y} = 0 \Rightarrow e^{2y} - 2xe^y - 1 = 0$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2(x \pm \sqrt{x^2+1})}{2} \Rightarrow e^y = x \pm \sqrt{x^2+1}$$

since $e^y > 0 \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \ln(e^y) = \ln(x + \sqrt{x^2+1})$

$$\Rightarrow y = \ln(x + \sqrt{x^2+1}) \Rightarrow \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

3) Let

$y = \tanh^{-1}(x) \Rightarrow x = \tanh(y) \Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow e^y - e^{-y} = xe^y + xe^{-y}$

$$\Rightarrow e^y - e^{-y} - xe^y - xe^{-y} = 0 \Rightarrow (1-x)e^y - (1+x)e^{-y} = 0$$

$$\Rightarrow (1-x)e^{2y} - (1+x) = 0 \Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow \ln(e^{2y}) = \ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \Rightarrow \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

18. Limits

If the values of a function $f(x)$ approach the value L as x approaches c, we say f has limit L as x approaches c and we write $\lim_{x\to c} f(x) = L$

Example (18.1): Find $\lim_{x\to -2} \frac{4}{x^2}$

$$\lim_{x \to -2} \frac{4}{x^2} = \frac{4}{(-2)^2} = 1$$

x	-2.1	-2.01	-2.001	-2.0001	⋯	-2	⋯	-1.999	-1.99	-1.9
$f(x)$	0.90702	0.99007	0.99900	0.99990	⋯	1	⋯	1.0010	1.0100	1.1080

left side → ← right side

Theorem (18.1):

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then

1) $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$

2) $\lim_{x \to a}(f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = A \times B$

3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}$, $B \neq 0$

4) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = kA$, k is constant

5) $\lim_{x \to a} k = k$, where k is constant

$\lim_{x \to a} x = a$

$\lim_{x \to a} x^n = \left(\lim_{x \to a} x\right)^n = a^n$

$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A}$, $A > 0$ if n

6)

7)

8)is even

Example (18.2): Find $\lim_{x \to 5}(x^2 - 4x + 3)$

Solution:

$$\lim_{x\to 5}(x^2-4x+3)=\lim_{x\to 5}x^2-\lim_{x\to 5}4x+\lim_{x\to 5}3=\lim_{x\to 5}x^2-4\lim_{x\to 5}x+\lim_{x\to 5}3=25-20+3=8$$

Example (18.3): Find $\lim_{x\to 2}\dfrac{5x^3+4}{x-3}$

Solution:

$$\lim_{x\to 2}\frac{5x^3+4}{x-3}=\frac{\lim_{x\to 2}5x^3+4}{\lim_{x\to 2}x-3}=\frac{40+4}{2-3}=\frac{44}{-1}=-44$$

Example (18.4): Find $\lim_{x\to 5}\dfrac{x^2-25}{x-5}$

Solution:

$$\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}}=\lim_{x\to 5}x+5=5+5=10$$

Exercises (18): Find the following limits:

1) $\lim_{x\to 1}\dfrac{x^3-1}{x-1}$ 2) $\lim_{x\to 0}\dfrac{5x^2-4}{x+1}$ 3) $\lim_{x\to 4}\dfrac{x^2-x-12}{x-4}$

4) $\lim_{x\to 3}\dfrac{x^2-6x+9}{x-3}$ 5) $\lim_{x\to 0}\dfrac{x}{\sqrt{x+1}-1}$ 6) $\lim_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{4}{x^2-4}\right)$

18.1 Right-Hand and Left-Hand Limits:

Let $f(x)$ be a function then the right-hand limit defined as $\lim_{x\to a^+}f(x)$ (the limit of $f(x)$ as x approaches a from the right). and the left-hand limit defined as $\lim_{x\to a^-}f(x)$ (the limit of $f(x)$ as x approaches a from the left).

Remark (18.1):

$\lim_{x\to a}f(x)=L$ if and only if $\lim_{x\to a^-}f(x)=L=\lim_{x\to a^+}f(x)$

Example (18.5): Find $\lim_{x\to 3}[x]$

Solution: $\lim_{x\to 3^-}[x] = 2$ and $\lim_{x\to 3^+}[x] = 3 \Rightarrow \lim_{x\to 3^-}[x] \neq \lim_{x\to 3^+}[x]$

$\therefore$ the limit dose not exists.

Example (18.6): If $f(x) = \begin{cases} 4 - x^2 & \text{if } x \le 1 \\ 2 + x^2 & \text{if } x > 1 \end{cases}$ find $\lim_{x\to 1} f(x)$

Solution:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(4 - x^2) = 4 - 1 = 3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2 + x^2) = 2 + 1 = 3$$

$$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 3 \Rightarrow \lim_{x\to 1} f(x) = 3$$

Theorem (18.2):

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

Example (18.7): Find $\lim_{x\to 0}\frac{1-\cos(x)}{x}$

Solution:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}\times\frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0}\frac{\sin(x)}{x}\times\lim_{x\to 0}\frac{\sin(x)}{1+\cos(x)} = 0$$

Example (18.8): Find $\lim_{x\to 0}\frac{\sin(15x)}{7x}$

Solution:

$$\lim_{x\to 0}\frac{\sin(15x)}{7x}\times\frac{15}{15} = \frac{15}{7}\lim_{x\to 0}\frac{\sin(15x)}{15x} = \frac{15}{7}\times 1 = \frac{15}{7}$$

Exercises (18.1): Find the following limits:

1) $\lim\limits_{x\to 0}\dfrac{\sin(x)}{\sqrt{x}}$ 2) $\lim\limits_{x\to 0} x\cot(x)$ 3) $\lim\limits_{y\to 0}\dfrac{1-\cos(y)}{y^2}$ 4) $\lim\limits_{t\to 0}\dfrac{\tan(t)}{2t}$ 5) $\lim\limits_{x\to 0}\dfrac{2x+1-\cos(x)}{3x}$ 6) $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{\sin(4x)}$

Theorem (18.3):

1) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

2) $\lim\limits_{x\to 0}\left(1+\dfrac{1}{x}\right)^{x} = e$

3) $\lim\limits_{x\to 0}(1+\lambda x)^{\frac{1}{x}} = e^{\lambda}\ ,\ \lambda$ any constant

4) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x} = 1$

5) $\lim\limits_{x\to 1}\dfrac{x-1}{\ln(x)} = 1$

6) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \ln(a)\ ,\ a$ any constant

7) $\lim\limits_{x\to 0}\dfrac{(1+x)^{\alpha}-1}{x} = \alpha$

Remark (18.2):

$$\lim_{x\to a}\log_c f(x) = \log_c \lim_{x\to a} f(x)$$

Example (18.9): Find the following limits

1) $\lim_{x\to 0}\frac{\log_a(1+x)}{x}$ 2) $\lim_{x\to 0}\frac{e^{2x}-e^{-3x}}{x}$

Solution:

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\log_a(1+x) = \lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}} = \log_a \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$
$$= \log_a e = \frac{\ln(e)}{\ln(a)} = \frac{1}{\ln(a)}$$

$$\lim_{x\to 0}\frac{e^{2x}-e^{-3x}+1-1}{x} = \lim_{x\to 0}\frac{e^{2x}-1}{x} - \lim_{x\to 0}\frac{e^{-3x}-1}{x} = 2\lim_{x\to 0}\frac{e^{2x}-1}{2x} + 3\lim_{x\to 0}\frac{e^{-3x}-1}{-3x}$$

1)

2)

= 2 × 1 + 3 × 1 = 5

Exercises (18.2): Find the following limits:

1) $\lim_{x\to 0}\frac{6^x-2^x}{x}$ 2) $\lim_{x\to a}\frac{x^a-a^x}{x-a}$ 3) $\lim_{x\to 0}\frac{a^{-x}-1}{x}$ 4) $\lim_{x\to 0}\frac{\cosh(x)-1}{x}$

18.2 Limits at Infinity

~ We say that $\lim_{x\to+\infty} f(x) = L$ if for any positive number we can find a positive number N such that $|f(x)-L|<\epsilon$ for all $x > N$.

~ We say that $\lim_{x\to-\infty} f(x) = L$ if for any positive number we can find a positive number N such that $|f(x)-L|<\epsilon$ for all $x < -N$.

Example (18.10): Prove that $\lim_{x\to\infty}\frac{2x}{3x+1}=\frac{2}{3}$

Solution:

$$|f(x)-L|<\epsilon \Rightarrow \left|\frac{2x}{3x+1}-\frac{2}{3}\right|<\epsilon \Rightarrow \left|\frac{6x-6x-2}{3(3x+1)}\right|<\epsilon \Rightarrow \left|\frac{-2}{9x+3}\right|<\epsilon \Rightarrow \frac{2}{9x+3}<\epsilon$$

$$\Rightarrow \frac{9x+3}{2}>\frac{1}{\epsilon} \Rightarrow 9x+3>\frac{2}{\epsilon} \Rightarrow 9x>\frac{2}{\epsilon}-3 \Rightarrow 9x>\frac{2-3\epsilon}{\epsilon} \Rightarrow x>\frac{2-3\epsilon}{9\epsilon}$$

Let $N=\dfrac{2-3\epsilon}{9\epsilon}$

$$\Rightarrow \lim_{x\to\infty}\frac{2x}{3x+1}=\frac{2}{3}$$

Theorem (18.4):

1) $\lim_{x\to\infty}\frac{1}{x}=0$ 2) $\lim_{x\to-\infty}\frac{1}{x}=0$

Theorem (18.5):

1) $\lim_{x\to+\infty} x^n=+\infty\ ,\ n=1,2,3,\cdots$

2) $\lim_{x\to-\infty} x^n = \begin{cases} -\infty & n=1,3,5,\cdots \\ +\infty & n=2,4,6,\cdots \end{cases}$

Example (18.11): Find $\lim_{x\to-\infty}\dfrac{x^2}{2x^2+1}$

Solution:

$$\lim_{x\to-\infty}\frac{\frac{x^2}{x^2}}{\frac{2x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{1}{2+\frac{1}{x^2}}=\frac{1}{2+\frac{1}{\infty}}=\frac{1}{2+0}=\frac{1}{2}$$

Example (18.12): Find $\lim_{x\to\infty}\sqrt[3]{\dfrac{3x+5}{6x-8}}$

Solution:

$$\lim_{x\to\infty}\sqrt[3]{\frac{3x+5}{6x-8}}=\sqrt[3]{\lim_{x\to\infty}\frac{3x+5}{6x-8}}=\sqrt[3]{\lim_{x\to\infty}\frac{3+\frac{5}{x}}{6-\frac{8}{x}}}=\sqrt[3]{\frac{1}{2}}$$

Exercises (18.3): Find the following limits:

1) $\lim_{x\to\infty}(\sqrt{x^6+5}-x^3)$ 2) $\lim_{x\to\infty}\frac{7x-4}{\sqrt{x^3+5}}$ 3) $\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}$

4) $\lim_{x\to\infty}\frac{\sqrt{5x^2-2}}{x+3}$ 5) $\lim_{x\to-\infty}-4x^8$

Theorem (18.6): If $g(x) \le f(x) \le h(x)$ for all x such that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, where L is constant

then $\lim_{x\to\infty} f(x) = L$

Example (18.13): Prove that $\lim_{x\to\infty}\frac{\sin(x)}{x} = 0$

Proof:

Since $-1 \le \sin(x) \le 1 \Rightarrow \frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$\because \lim_{x\to\infty}\frac{-1}{x} = 0$ and $\lim_{x\to\infty}\frac{1}{x} = 0$

$\therefore \lim_{x\to\infty}\frac{\sin(x)}{x} = 0$

Example (18.14): Find $\lim_{x\to\infty}\frac{\cos^2(2x)}{4x^2}$ (H.W)

Example (18.15): Find $\lim_{x\to-\infty}\left(1+\frac{2}{x}\right)\cos\left(\frac{1}{x}\right)$

Solution:

$$\lim_{x\to-\infty}\left(1+\frac{2}{x}\right)\cos\left(\frac{1}{x}\right) = 1$$

Example (18.16): Find $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Solution:

Let $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$, at $x \to \infty$ then $y \to 0$

$$\therefore \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{y\to 0}\frac{\sin(y)}{y} = 1$$

19. Continuity

Definition (19.1): A function *f* is said to be continuous at $x = c$ provided the following conditions are satisfied:

i. $f(c)$ is defined

ii. $\lim_{x \to c} f(x)$ exists

iii. $\lim_{x \to c} f(x) = f(c)$

Example (19.1): Determine whether the following functions are continuous or not at $x = 2$.

1) $f(x) = \dfrac{x^2 - 4}{x - 2}$

2) $g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \text{ 6= } 2 \\ 3 & \text{if } x = 2 \end{cases}$

3) $h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \text{ 6= } 2 \\ 4 & \text{if } x = 2 \end{cases}$

Solution:

1) $f(2) = \dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$ not defined

$\therefore f(x)$ is discontinuous

2)

i. $g(2) = 3$

ii. $\lim_{x \to 2} g(x) = \lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2} \dfrac{(x - 2)(x + 2)}{x - 2}$ = 4 exists

iii. $\lim_{x \to 2} g(x) \neq g(2)$

$\therefore g(x)$ is discontinuous

3)

i. $h(2) = 4$

ii. $\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$ = 4 exist

iii. $\lim_{x \to 2} h(x) = h(2)$

$\therefore h(x)$ is continuous

Theorem (19.1):

Every polynomial functions are continuous.

Theorem (19.2):

A rational functions are continuous at every number where the denominator is non zero.

Theorem (19.3):

If the functions *f* and *g* are continuous at *c*, then:

1) $f \mp g$ is continuous at *c*
2) *f.g* is continuous at *c*

3) *f/g* is continuous at *c* if *g*(*c*) 6= 0

Example (19.2): Show that whether the function $f(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$ continuous or not? **Solution:**

$x^2 - 5x + 6 = 0 \Rightarrow (x - 3)(x - 2) = 0 \Rightarrow x = 3, x = 2$

$\therefore f(x)$ continuous at every points except $x = 3$ and $x = 2$

Exercises (19): Show that whether the following functions are continuous or not?

1) $g(x) = |x|$ at $x = 0$

2) $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ 3x + 1 & \text{if } x < 1 \end{cases}$ at $x = 1$

Theorem (19.4):

The functions $\sin(x)$ and $\cos(x)$ are continuous functions.

Theorem (19.5):

i. If the function $g(x)$ is continuous at c, and $f(x)$ continuous at $g(c)$, then $f \circ g$ is continuous at c.

ii. If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Example (19.3): Show that the function $h(x) = \left(\frac{x\sin(x)}{x^2+2}\right)^2$ is continuous at every value of x.

Solution: $f(x) = x^2$ and

$g(x) = \frac{x\sin(x)}{x^2+2}$

$g_1(x) = \frac{x}{x^2+2}$ and $g_2(x) = \sin(x)$

∵ $f(x)$ is continuous (by **Theorem (19.1)**) Since

$g^1(x)$ is continuous (by **Theorem (19.2)**) and

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[1]) $y = x^2 + 2x + 1$ 2) $y = \sqrt{x^2 + 3}$

Solution: 1

$g_2(x)$ is continuous (by **Theorem (19.4)**) $\therefore g(x)$

is continuous (by **Theorem (19.3)**)

$\therefore (f \circ g)(x) = \left(\frac{x\sin(x)}{x^2+2}\right)^2$ is continuous (by **Theorem (19.5)**)

$\therefore h(x)$ is continuous.

20. Derivative:

The derivative of a function f is the function f^0 whose value at x is defined by the

equation:

$$\frac{df}{dx} = \frac{d}{dx}f(x) = \frac{dy}{dx} = y' = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Definition (20.1):

A function that has a derivative at a point x is said to be differentiable at x.

Definition (20.2):

A function that is differentiable at every point of its domain is called differentiable.

Definition (20.3):

When the number $f^0(x)$ exists it is called the slope of the curve $y = f(x)$ at x.

The line through the point $(x, f(x))$ with slope $f^0(x)$ is the tangent to the curve at x.

Example (20.1): Find $\frac{dy}{dx}$ by definition for the following functions:

$$y' = \frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2+2(x+h)+1-x^2-2x-1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h + \cancel{1} - \cancel{x^2} - \cancel{2x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} = 2x + 2$$

Solution: 2

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + 3} - \sqrt{x^2 + 3}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3 - (x^2 + 3)}{h(\sqrt{(x+h)^2 + 3} + \sqrt{x^2 + 3})} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3} - \cancel{x^2} - \cancel{3}}{h(\sqrt{(x+h)^2 + 3} + \sqrt{x^2 + 3})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}(\sqrt{(x+h)^2 + 3} + \sqrt{x^2 + 3})} = \frac{2x}{\sqrt{x^2 + 3} + \sqrt{x^2 + 3}} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2 + 3}} = \frac{x}{\sqrt{x^2 + 3}}$$

Exercises (20.1): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = x^3 +$ 3   2) $\sqrt{\_}$ 3) $y = \frac{x+1}{x-1}$ 4) $y = \frac{1}{x}$ 5) $y = \frac{1}{\sqrt{x+1}}$

Differentiation Theorem:

1) $\frac{d}{dx}(c) = 0$, c is constant.

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f(x) \mp g(x)) = \frac{d}{dx}(f(x)) \mp \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(x) \times g(x)) = f(x) \times \frac{d}{dx}(g(x)) + g(x) \times \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \times \frac{d}{dx}(f(x)) - f(x) \times \frac{d}{dx}(g(x))}{(g(x))^2} \text{ , } g(x) \neq 0$$

$$\frac{d}{dx}(f(x))^n = n \times (f(x))^{n-1} \times \frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2)

3)

4)

5)

6)

7)

Example (20.2): Find f' of the following functions:

1) $f(x) = x + \frac{1}{x^2} + 3$ 2) $f(x) = \sqrt{x^3-2} + \frac{1}{\sqrt{x+1}}$ 3) $f(x) = (x^2+1)^3(x^3-1)^2$

Solution: 1

$$f'(x) = 1 - \frac{2x}{x^4} = 1 - \frac{2}{x^3}$$

Solution: 2

$$f(x) = (x^3-2)^{\frac{1}{2}} + (x+1)^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}(x^3-2)^{-\frac{1}{2}} \times 3x^2 - \frac{1}{2}(x+1)^{-\frac{3}{2}} \times 1 = \frac{3}{2}\frac{x^2}{\sqrt{x^3-2}} - \frac{1}{2}\frac{1}{\sqrt{(x+1)^3}}$$

Solution: 3

$f'(x) = (x^2+1)^3 \times 2(x^3-1) \times 3x^2 + (x^3-1)^2 \times 3(x^2+1)^2 \times 2x$ =

$6x^2(x^2+1)^3(x^3-1) + 6x(x^3-1)^2(x^2+1)^2$

Exercises (20.2): Find f' of the following functions:

1) $f(x) = \left(\frac{x+1}{x^2-2}\right)^3$ 2) $f(x) = x^2 + \frac{1}{x^2}$ 3) $f(x) = \frac{x^2+1}{x^2-1}$, $x^2 \neq 1$

4) $f(x) = (x-1)^3(x+2)^4$ 5) $f(x) = \frac{x^3-1}{\sqrt{x+1}}$ 6) $f(x) = (x^2+1)^8$

7) $f(x) = (x+1)^2(x^2+1)^{-3}$

20.1 Second and Higher-Order Derivative:

If the derivative f^0 of a function f itself differentiable then the derivative of f^0 is denoted by f^{00} and is called the second derivative of f.

$$i.e:\ f'(x) = \frac{d}{dx}(f(x))$$

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \frac{d}{dx}\left[\frac{d}{dx}(f(x))\right]$$

$$f'''(x) = \frac{d^3}{dx^3}(f(x)) = \frac{d^2}{dx^2}\left[\frac{d}{dx}(f(x))\right]$$

$$\vdots$$

$$f^{(n)}(x) = \frac{d^n}{dx^n}(f(x))$$

Example (20.3): Find $f^{(5)}(x)$ where $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$

Solution:

$f^0(x) = 12x^3 - 6x^2 + 2x - 4$

$f^{00}(x) = 36x^2 - 12x + 2\ f^{000}(x)$

$= 72x - 12\ f^{(4)}(x) = 72\ f^{(5)}(x)$

$= 0$

Exercises (20.3): Find $\frac{d^4y}{dx^4}$ where $y = \frac{3}{x^3}$

Theorem (20.1):

If f has a derivative at $x = c$, then f is continuous at c.

20.2 Chain Rule:

i. If *y* is a differentiable function of *u* and *u* is a differentiable function of *x* then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

ii. If *y* is a differentiable function of *u* and *x* is a differentiable function of *u* then,

$$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

Example (20.4): If $y = t^4 + 2t + 3$, $x = t^2 + 1$ find $\frac{dy}{dx}$

Solution:

$\frac{dy}{dt} = 4t^3 + 2$ and $\frac{dx}{dt} = 2t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3+2}{2t} = \frac{2t^3+1}{t} = \frac{2(\sqrt{x-1})^3+1}{\sqrt{x-1}} = \frac{2(x-1)^{\frac{3}{2}}+1}{\sqrt{x-1}}$$

Example (20.5): If $y = \frac{u^3+1}{u^3-2}$, $u = \sqrt{x} + 1$ find $\frac{dy}{dx}$

Solution:

$$\frac{dy}{du} = \frac{(u^3-2).3u^2 - (u^3+1).3u^2}{(u^3-2)^2} = \frac{\cancel{3u^5} - 6u^2 - \cancel{3u^5} - 3u^2}{(u^3-2)^2} = \frac{-9u^2}{(u^3-2)^2}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-9u^2}{(u^3-2)^2} \times \frac{1}{2\sqrt{x+1}} = \frac{-9\cancel{(x+1)}^{\sqrt{x+1}}}{\left((x+1)^{\frac{3}{2}}-2\right)^2} \times \frac{1}{2\cancel{\sqrt{x+1}}}$$

$$= \frac{-9\sqrt{x+1}}{2\left((x+1)^{\frac{3}{2}}-2\right)^2}$$

Exercises (20.4):

1) If $y = t^2 + 2t$, $t = \frac{x-2}{3-x}$ find $\frac{dy}{dx}$

2) If $y = \sqrt{t} + \frac{1}{\sqrt{t}}$, $x = t^2 + 2t$ find $\frac{dy}{dx}$

3) If $x = \sqrt{t} - t^3$, $y = t^{\frac{2}{3}} + t^2$ find $\frac{dy}{dx}$

4) If $y = s^2$, $s = r + 1$, $r = t^2 - 5$, $t = w + 3$, $w = x^2$ find $\frac{dy}{dx}$

20.3 Implicit differentiation:

If y can not be written in the form $y = f(x)$ then to find $\frac{dy}{dx}$:

i. Differentiable both sides with respect to x. ii.

Solve the result for $\frac{dy}{dx}$.

Example (20.6): Find $\frac{dy}{dx}$ for the functions

1) $x^3 + y^3 = 3xy$

Solution:

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y \Rightarrow 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$
$$\Rightarrow \frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

2) $xy + y^2x + 3y - 2x = 0$

Solution:

$$\Rightarrow x\frac{dy}{dx} + y + y^2 + 2yx\frac{dy}{dx} + 3\frac{dy}{dx} - 2 = 0$$
$$\therefore \frac{dy}{dx} = \frac{2 - y - y^2}{x + 2xy + 3}$$

3) $\frac{1}{yx^2} + \frac{1}{yx} = y + x$

Solution:

$$\Rightarrow (yx^2)^{-1} + (yx)^{-1} = y + x$$
$$\Rightarrow -(yx^2)^{-2}\left(2yx + x^2\frac{dy}{dx}\right) - (yx)^{-2}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx} + 1$$
$$\Rightarrow -2yx(yx^2)^{-2} - x^2(yx^2)^{-2}\frac{dy}{dx} - y(yx)^{-2} - x(yx)^{-2}\frac{dy}{dx} = \frac{dy}{dx} + 1$$
$$\Rightarrow \frac{dy}{dx}\left(-x^2(yx^2)^{-2} - x(yx)^{-2} - 1\right) = 2yx(yx^2)^{-2} + y(yx)^{-2} + 1$$
$$\therefore \frac{dy}{dx} = \frac{2yx(yx^2)^{-2} + y(yx)^{-2} + 1}{-x^2(yx^2)^{-2} - x(yx)^{-2} - 1}$$

Exercises (20.5): Find $\frac{dy}{dx}$ if

$x^2y^2 + \frac{x}{y} = 0$ 2) $\frac{x^2y}{x-y} = \frac{3x}{4+y}$ 1) –

$\frac{1}{x} + \frac{1}{y} = 1$ 4) $xy^2 = \frac{x+y}{x-y}$

$y = \sqrt{\sqrt{x} + \sqrt{x^2 + \sqrt{x}}}$

3)

5)

20.4 Derivatives of Trigonometric Functions:

1) $\frac{d}{dx}(\sin(u)) = \cos(u).\frac{du}{dx}$

$$\frac{d}{dx}(\cos(u)) = -\sin(u).\frac{du}{dx}$$
$$\frac{d}{dx}(\tan(u)) = \sec^2(u).\frac{du}{dx}$$
$$\frac{d}{dx}(\cot(u)) = -\csc^2(u).\frac{du}{dx}$$

2)

3)

4)

5) $\frac{d}{dx}(\sec(u)) = \sec(u)\tan(u).\frac{du}{dx}$

6) $\frac{d}{dx}(\csc(u)) = -\csc(u)\cot(u).\frac{du}{dx}$

Example (20.7): Find $\frac{dy}{dx}$ or $f'(x)$ if

1) $f(x) = \tan(3x^2)$ **Solution:**

$f^0(x) = 6x\sec^2(3x^2)$

2) $y = \sin(2x) + \sec(3x)$

Solution:

$$\Rightarrow \frac{dy}{dx} = 2\cos(2x) + 3\sec(3x)\tan(3x)$$

3) $y = \cos(\sqrt{x})$

Solution:

$$\Rightarrow \frac{dy}{dx} = -\sin(\sqrt{x}).\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$$

4) $y^2 = x^2 + \sin(xy)$

Solution:

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(x\frac{dy}{dx} + y\right) \Rightarrow 2y\frac{dy}{dx} = 2x + x\cos(xy)\frac{dy}{dx} + y\cos(xy)$$

$$\Rightarrow 2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\therefore \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

5) $xy = \csc(x - y)$

Solution:

$$x\frac{dy}{dx} + y = -\csc(x-y)\cot(x-y)\left(1 - \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-y - \csc(x-y)\cot(x-y)}{x - \csc(x-y)\cot(x-y)}$$

Exercises (20.6): Find $\frac{dy}{dx}$ for the following functions:

1) $y^2x = \cos^3(x-y)^2$

2) $y = x^2\tan(x^2)$

3) $y = \cot\left(\frac{\sin^2(x)}{\tan(x)}\right)$

4) $yx^2 = \sin^4(x^3)$

5) $y = \tan^2(x)\cot^2(1-x)$

6) $y = \tan^2(x)\cot^2(x)$

20.5 Derivatives of the Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$$
$$\frac{d}{dx}(\cos^{-1}(u)) = \frac{-1}{\sqrt{1-u^2}}.\frac{du}{dx}$$
$$\frac{d}{dx}(\tan^{-1}(u)) = \frac{1}{1+u^2}.\frac{du}{dx}$$
$$\frac{d}{dx}(\cot^{-1}(u)) = \frac{-1}{1+u^2}.\frac{du}{dx}$$

1)

2)

3)

4)

$$\frac{d}{dx}(\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$$
$$\frac{d}{dx}(\csc^{-1}(u)) = \frac{-1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$$

5)

6)

Proof: 1

$$\because \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $y = \sin^{-1}(u) \Rightarrow \sin(y) = u$

$$\Rightarrow \cos(y).\frac{dy}{dx} = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}.\frac{du}{dx}$$

$$\because \sin(y) = u \Rightarrow \sin^2(y) = u^2 \Rightarrow 1 - \sin^2(y) = 1 - u^2 \Rightarrow \cos^2(y) = 1 - u^2$$

$$\Rightarrow \sqrt{\cos^2(y)} = \sqrt{1-u^2} \Rightarrow \cos(y) = \sqrt{1-u^2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$$

Example (20.8): Find $\frac{dy}{dx}$ if

1) $y = \sin^{-1}(3x^2)$ **Solution:**

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(3x^2)^2}}.6x = \frac{6x}{\sqrt{1-9x^4}}$$

2) $y = \tan^{-1}(3\tan(x))$

Solution:

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(3\tan(x))^2}.3\sec^2(x) = \frac{3\sec^2(x)}{1+9\tan^2(x)}$$

3) $y = \sec^{-1}(2x^2)$ **Solution:**

$$\Rightarrow \frac{dy}{dx} = \frac{1}{|2x^2|\sqrt{(2x^2)^2-1}}.4x = \frac{4x}{2x^2\sqrt{4x^4-1}}$$

Exercises (20.7): Find $\frac{dy}{dx}$ for the following functions:

1) $y = \tan^{-1}\left(\sqrt{x+1}\right)$

2) $y = x\cos^{-1}(3x)$

3) $y = \cot^{-1}\left(\frac{x}{2}\right) + \tan^{-1}\left(\frac{x}{2}\right)$

4) $y = \cot^{-1}\left(\frac{x+1}{1-x}\right)$

20.6 Derivatives of the Logarithmic and Exponential Functions:

1) $$\frac{d}{dx}(\ln(u)) = \frac{1}{u}.\frac{du}{dx}$$

2) $$\frac{d}{dx}(\log_a(u)) = \frac{1}{u\ln(a)}.\frac{du}{dx}$$

3) $$\frac{d}{dx}(a^u) = a^u\ln(a).\frac{du}{dx}$$

4) $$\frac{d}{dx}(e^u) = e^u.\frac{du}{dx}$$

Example (20.9): Find $\frac{dy}{dx}$ for the following functions:

1) $$y = \ln(x^3)$$ $$\Rightarrow \frac{dy}{dx} = \frac{1}{x^3}.3x^2 = \frac{3}{x}$$

2) $$y = \ln(\sin^{-1}(2x))$$ $$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin^{-1}(2x)}.\frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sin^{-1}(2x)\sqrt{1-4x^2}}$$

3) $$y = (100)^{x^2+2x}$$ $$\Rightarrow \frac{dy}{dx} = (100)^{x^2+2x}\ln(100).(2x+2)$$

4) $$y = e^{\sin(x)}$$ $$\Rightarrow \frac{dy}{dx} = e^{\sin(x)}\cos(x) = \cos(x)e^{\sin(x)}$$

$$y = x\log_3 x$$
$$\Rightarrow \frac{dy}{dx} = x.\left(\frac{1}{x\ln(3)}\right) + \log_3 x = \frac{1}{\ln(3)} + \log_3 x$$

$$y = e^{\ln(x)+x}$$
$$\Rightarrow \frac{dy}{dx} = e^{\ln(x)+x}.\left(\frac{1}{x}+1\right) = e^{\ln(x)}e^{x}\left(\frac{1+x}{x}\right)$$
$$= xe^{x}\left(\frac{1+x}{x}\right) = e^{x}(1+x)$$

5)

6)

Exercises (20.8): Find $\frac{dy}{dx}$ for the following functions:

1) $y = e^{\ln(x)-\ln(1+x)}$ 2) $y = \ln\left(\frac{1}{x}\right)$ 3) $y = e^{\ln\left(\frac{1}{x^2}\right)}$ 4) $y = \frac{\log_3 x^2}{\log_2 x}$

Example (20.10): Find $\frac{dy}{dx}$ for the following functions:

1) $y = (\sin(x))^{\cos(x)}$

Solution:

$$\Rightarrow \ln(y) = \ln(\sin(x))^{\cos(x)}$$
$$\Rightarrow \ln(y) = \cos(x)\ln(\sin(x))$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos(x).\frac{1}{\sin(x)}.\cos(x) + \ln(\sin(x)).(-\sin(x))$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos(x)\cot(x) - \sin(x)\ln(\sin(x))$$
$$\Rightarrow \frac{dy}{dx} = y(\cos(x)\cot(x) - \sin(x)\ln(\sin(x)))$$
$$= (\sin(x))^{\cos(x)}(\cos(x)\cot(x) - \sin(x)\ln(\sin(x)))$$

2) $y^x = x^y$

Solution:

$$\ln(y^x) = \ln(x^y) \Rightarrow x\ln(y) = y\ln(x) \Rightarrow x.\frac{1}{y}\frac{dy}{dx} + \ln(y) = y\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

$$\frac{x}{y}\frac{dy}{dx} - \ln(x)\frac{dy}{dx} = \frac{y}{x} - \ln(y) \Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x} - \ln(y)}{\frac{x}{y} - \ln(x)}$$

Exercises (20.9): Find $\frac{dy}{dx}$ for the following functions:

1) $y = (x)^{\sin(x)}$ 2) $y = x^x$ 3) $y = x^{x^2}$

20.7 Derivatives of Hyperbolic Functions:

1) $$\frac{d}{dx}(\sinh(u)) = \cosh(u).\frac{du}{dx}$$

2) $$\frac{d}{dx}(\cosh(u)) = \sinh(u).\frac{du}{dx}$$

3) $$\frac{d}{dx}(\tanh(u)) = \text{sech}^2(u).\frac{du}{dx}$$

4) $$\frac{d}{dx}(\coth(u)) = -\text{csch}^2(u).\frac{du}{dx}$$

5) $$\frac{d}{dx}(\text{sech}(u)) = -\text{sech}(u)\tanh(u).\frac{du}{dx}$$

6) $$\frac{d}{dx}(\text{csch}(u)) = -\text{csch}(u)\coth(u).\frac{du}{dx}$$

Proof:

1

$$y = \sinh(u) = \frac{e^u - e^{-u}}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^u.\frac{du}{dx} + e^{-u}.\frac{du}{dx}\right) = \frac{1}{2}\left(e^u + e^{-u}\right)\frac{du}{dx} = \cosh(u).\frac{du}{dx}$$

Example (20.11): Find $\frac{dy}{dx}$ for the following functions:

1)
$$y = \sinh^2(5x)$$
$$\Rightarrow \frac{dy}{dx} = 2\sinh(5x).\cosh(5x).5$$
$$\Rightarrow \frac{dy}{dx} = 10\sinh(5x)\cosh(5x)$$

2) $y = \tanh(x^3)\coth(x^2)$
$$\Rightarrow \frac{dy}{dx} = \tanh(x^3)(-\operatorname{csch}^2(x^2)).(2x) + \coth(x^2)(\operatorname{sech}^2(x^3)).(3x^2)$$

3)
$$y = \cosh(e^{2x})$$
$$\Rightarrow \frac{dy}{dx} = \sinh(e^{2x}).e^{2x}.2 = 2e^{2x}\sinh(e^{2x})$$

4)
$$y = \ln(\sinh(2x))$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh(2x)}.\cosh(2x).2 = \frac{2\cosh(2x)}{\sinh(2x)} = 2\coth(2x)$$

Exercises (20.10): Find $\frac{dy}{dx}$ for the following functions:

1) $y = \operatorname{sech}^3(2x)$ 2) $y = \sinh(\tan(x))$ 3) $y = \cosh\left(xe^{\sinh(x)}\right)$

20.8 Derivatives of the Inverse Hyperbolic Functions:

1)
$$\frac{d}{dx}(\sinh^{-1}(u)) = \frac{1}{\sqrt{u^2+1}}.\frac{du}{dx}$$

2)
$$\frac{d}{dx}(\cosh^{-1}(u)) = \frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}$$

3)
$$\frac{d}{dx}(\tanh^{-1}(u)) = \frac{1}{1-u^2}.\frac{du}{dx}$$

$$d \quad {}^{-1}(u)) = \frac{1}{1-u^2}.\frac{du}{dx}$$ 4)—(coth ,

$|u| > 1$ dx

$$\frac{d}{dx}(\text{sech}^{-1}(u)) = \frac{-1}{u\sqrt{1-u^2}}.\frac{du}{dx}$$

$$\frac{d}{dx}(\text{csch}^{-1}(u)) = \frac{-1}{|u|\sqrt{1+u^2}}.\frac{du}{dx}$$

5)

6)

Proof: 1

Let $y = \sinh^{-1}(u) = \ln(u + \sqrt{u^2+1})$

$$\because \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u+\sqrt{u^2+1}}.\left(1+\frac{2u}{2\sqrt{u^2+1}}\right).\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u+\sqrt{u^2+1}}.\left(\frac{u+\sqrt{u^2+1}}{\sqrt{u^2+1}}\right).\frac{du}{dx} = \frac{1}{\sqrt{u^2+1}}.\frac{du}{dx}$$

Example (20.12): Find $\frac{dy}{dx}$ for the following functions:

1)
$$y = \tanh^{-1}(\cos(x))$$
$$\Rightarrow y' = \frac{1}{1-\cos^2(x)}.(-\sin(x)) = \frac{-\sin(x)}{1-\cos^2(x)} = \frac{-1}{\sin(x)}$$

2)

$$y = \text{sech}^{-1}(\sin(2x))$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin(2x)\sqrt{1-\sin^2(2x)}}.(2\cos(2x)) = \frac{-2\cos(2x)}{\sin(2x)\sqrt{\cos^2(2x)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2\cos(2x)}{\sin(2x)\cos(2x)} = \frac{-2}{\sin(2x)}$$

$$y = \cosh^{-1}(e^x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{(e^x)^2 - 1}}.e^x = \frac{e^x}{\sqrt{e^{2x} - 1}}$$

$$y = \text{sech}^{-1}(\cos(x))$$
$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\cos(x)\sqrt{1 - \cos^2(x)}}.(-\sin(x)) = \sec(x)$$

3)

4)

5)
$$y = e^{\text{csch}^{-1}(x) + \coth^{-1}(x)}$$
$$\Rightarrow \frac{dy}{dx} = e^{\text{csch}^{-1}(x) + \coth^{-1}(x)}. \left(\frac{-1}{|x|\sqrt{1 + x^2}} + \frac{1}{1 - x^2} \right)$$

Exercises (20.11): Find $\frac{dy}{dx}$ for the following functions:

1) $y = \coth^{-1}\left(\frac{1}{x}\right)$ 2) $y = e^{\tanh^{-1}(2x)}$ 3) $y = \ln\left(\coth^{-1}\left(e^{\sin(x)}\right)\right)$

21. L'Hˆopital's Rule:

Suppose that $f(x_\circ) = g(x_\circ)$ and that the functions f and g are both differentiable on an open interval (a,b) that contains the point $x_\circ$.

Suppose also g^0 6= 0 at every point in (a,b) except possibly $x_\circ$, then

$$\lim_{x \to x_\circ} \frac{f(x)}{g(x)} = \lim_{x \to x_\circ} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

i. The Form $\left(\frac{0}{0} \;\&\; \frac{\infty}{\infty}\right)$

Example (21.1): Find $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0}$

Solution:

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{2}}-1}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1} = \frac{1}{2}$$

Example (21.2): Find $\lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} = \frac{0}{0}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} = \lim_{x\to 0} \frac{\sin(x)}{1+2x} = \frac{0}{1} = 0$$

Example (21.3): Find $\lim_{x\to 3} \frac{x^4-81}{x-3} = \frac{0}{0}$

Solution:

$$\lim_{x\to 3} \frac{x^4-81}{x-3} = \lim_{x\to 3} \frac{4x^3}{1} = 108$$

Example (21.4): Find $\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$

Solution:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Exercises (21.1): Find

1) $\lim_{x\to a} \frac{\sec(x)-\sec(a)}{x-a}$ 2) $\lim_{x\to 0} \frac{e^x-1}{x}$

3) $\lim_{\theta\to 0} \frac{\sin(2\theta)-2\sin(\theta)}{\sin(3\theta)-3\sin(\theta)}$ 4) $\lim_{x\to -\frac{\pi}{2}} \frac{\tan(x)}{1+\tan(x)}$

ii. The Form $(0.\infty \;\&\; \infty - \infty)$

Example (21.5): Find $\lim_{x\to\infty} x^2 e^{-x} = 0.\infty$

Solution:

$$\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Example (21.6): Find $\lim_{x\to 0} \left(\csc(x) - \frac{1}{x}\right) = \infty - \infty$

Solution:

$$\lim_{x\to 0}\left(\csc(x)-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin(x)}{x\sin(x)}\right)=\frac{0}{0}$$
$$=\lim_{x\to 0}\frac{1-\cos(x)}{x\cos(x)+\sin(x)}=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}=\frac{0}{2}=0$$

iii. The Form $0^0\ ,\ \infty^0\ ,\ 1^\infty)$

Example (21.7): Find $\lim_{x\to 0}(\cos(x))^{\frac{1}{x^2}}=1^\infty$

Solution:

Let $y=(\cos(x))^{\frac{1}{x^2}}\Rightarrow \ln(y)=\frac{1}{x^2}\ln(\cos(x))\Rightarrow \lim_{x\to 0}\ln(y)=\lim_{x\to 0}\frac{1}{x^2}\ln(\cos(x))$

$$=\lim_{x\to 0}\frac{-\sin(x)}{2x\cos(x)}=\lim_{x\to 0}\frac{-\cos(x)}{2(\cos(x)-x\sin(x))}=-\frac{1}{2}$$
$$\Rightarrow \lim_{x\to 0}\ln(y)=-\frac{1}{2}\Rightarrow \ln\left(\lim_{x\to 0}y\right)=-\frac{1}{2}$$
$$\Rightarrow e^{\ln\left(\lim_{x\to 0}y\right)}=e^{-\frac{1}{2}}\Rightarrow \lim_{x\to 0}y=e^{-\frac{1}{2}}$$
$$\therefore \lim_{x\to 0}(\cos(x))^{\frac{1}{x^2}}=e^{-\frac{1}{2}}$$

Example (21.8): Find $\lim_{x\to \frac{\pi}{2}}(\sin(x)-\cos(x))^{\tan(x)}=1^\infty$

Solution:

Let $y=(\sin(x)-\cos(x))^{\tan(x)}\Rightarrow \ln(y)=\tan(x)\ln(\sin(x)-\cos(x))$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\ln(y)=\lim_{x\to\frac{\pi}{2}}\tan(x)\ln(\sin(x)-\cos(x))$$
$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin(x)\ln(\sin(x)-\cos(x))}{\cos(x)}$$
$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin(x)\left(\frac{\cos(x)+\sin(x)}{\sin(x)-\cos(x)}\right)+\cos(x)\ln(\sin(x)-\cos(x))}{-\sin(x)}=-1$$
$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\ln(y)=-1\Rightarrow \lim_{x\to\frac{\pi}{2}}y=e^{-1}$$
$$\therefore \lim_{x\to\frac{\pi}{2}}(\sin(x)-\cos(x))^{\tan(x)}=\frac{1}{e}$$

Exercises (21.2): Prove that

1) $\lim_{x\to\frac{\pi}{2}}(\tan(x))^{\cos(x)}=1$ 2) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Exercises (21.3): Find

3) $\lim_{x\to\frac{\pi}{2}} (2x-\pi)\sec(x$ $\lim_{x\to 0}(1-x)^{\ln(x)}$ 1)) 2) $\lim_{x\to 1}(\sec^3(x))^{\cot_2(x)}$

22. Applications of Derivative:

i. Engineering Applications:

Example (22.1): Find the slope of the parabola $y = x^2$ at $x = 2$.

Solution:

$\because m = y^0 = 2x$

$\therefore m|_{x=2} = 2 \times 2 = 4$

Example (22.2): Find the equation for the tangent to the curve $y = \sqrt{x+1}$ at (1,2)

Solution:

$$y = (x+1)^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}$$
$$\therefore m|_{x=1} = \frac{1}{2\sqrt{2}}$$
$$\because y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 2 = \frac{1}{2\sqrt{2}}(x-1) \Rightarrow y = \frac{1}{2\sqrt{2}}(x-1) + 2$$

Remark (22.1):

$$\text{The slope for the normal} = \frac{-1}{\text{slope of tangent}}$$

Example (22.3): Find the equation for the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1,2)$

Solution:

$$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-2x}{2y-x}$$

$$\therefore m|_{(x,y)=(-1,2)} = \frac{4}{5}$$

∴ The slope of the normal= $\frac{-5}{4}$

$$\therefore y - y_1 = m(x - x_1) \Rightarrow y = \frac{-5}{4}(x+1) + 2$$

Exercises (22.1):

1) Find the equation for the tangent and normal to the curve $y = x^2 + 2x + 1$ at intersection point with vertical line ($y - axis$).

2) Find the equation for the tangent to the curve $y = -x^2 + 2x + 3$ at intersection point with horizontal line ($x - axis$).

ii. Physical Applications:

Definition (22.1): If $s(t)$ is the position function of a particle moving on a coordinate line, then the velocity of the particle at time t is defined by.

$$v(t) = \frac{ds}{dt}$$

Definition (22.2): If $s(t)$ is the position function of a particle moving on a coordinate line, then the acceleration of the particle at time t is defined by.

$$a(t) = \frac{dv}{dt} \quad \text{or} \quad a(t) = \frac{d^2s}{dt^2}$$

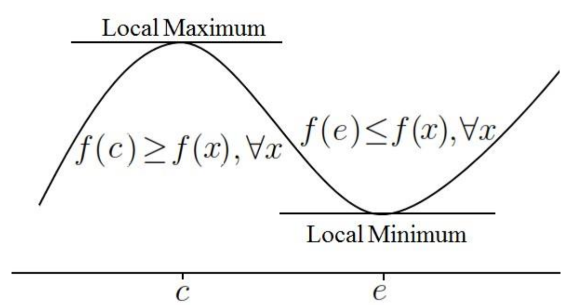
Example (22.4): Find the body's velocity and acceleration at time $t = 2$ if the position $s(t) = 4 + 2t + t^2$ of body moving along a coordinate line, where s is in meters and t is in seconds.

Solution:

$$v(t) = \frac{ds}{dt} = 2 + 2t \Rightarrow v|_{t=2} = 2 + 4 = 6 \ m/sec$$
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2 \ m/sec$$

23. Maximum, Minimum and Mean Values:

Definition (23.1): A function f has a local maximum value at an interior point c if $f(c) \geq f(x)$, $\forall x$. And f has a local minimum value at interior point e if $f(e) \leq f(x)$, $\forall x$



Theorem (23.1):

If a function f has a local maximum or local minimum value at point c and f^0 is defined at c, then $f^0(c) = 0$

Remark (23.1):

1. If $f^0(c) = 0$ and $f^{00}(c) < 0$, then f has local maximum at $x = c$
2. If $f^0(c) = 0$ and $f^{00}(c) > 0$, then f has local minimum at $x = c$

Rolle's Theorem:

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) and If $f(a) = f(b) = 0$ then there is at least one number c in (a,b) such that $f^0(c) = 0$.

Example (23.1): Find all values of c which satisfy *the Rolle's theorem* of the function

$f(x) = \frac{1}{3}x^3 - 3x$, $-3 \le x \le 3$

Solution:

The polynomial function $f(x) = \frac{1}{3}x^3 - 3x$ is continuous at every point of the interval $[-3,3]$ and differentiable at every point of the interval $(-3,3)$.

$$f(3) = \frac{27}{3} - 9 = 9 - 9 = 0$$

$$f(-3) = \frac{-27}{3} + 9 = -9 + 9 = 0$$

$$\therefore f(3) = f(-3) = 0$$

$\therefore$ By Rolle's Theorem, $\exists\ c \in (-3,3)$ 3 $f^0(c) = 0$

$$\therefore f'(x) = \frac{3}{3}x^2 - 3$$

$$\Rightarrow f'(c) = c^2 - 3 = 0 \Rightarrow c^2 = 3 \Rightarrow c = \mp\sqrt{3}$$

$\therefore$ There exists two numbers $c = \sqrt{3}$ and $c = -\sqrt{3}$ such that $f^0(\sqrt{3}) = 0$ and $f^0(-\sqrt{3}) = 0$

The Mean Value Theorem:

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example (23.2): Find all values of c which satisfy *the mean value theorem* for the following functions.

1) $\frac{1}{2} \le x \le 2$

$f(x) = x + \frac{1}{x}$, 2), $1 \le x \le 3$

3) $f(x) = 4 - x^2$, $-1 \le x \le 1$

$f(x) = \sqrt{x-1}$

Solution: 1

The function $f(x) = x + \frac{1}{x}$ is continuous on $\left[\frac{1}{2}, 2\right]$ and differentiable on $\frac{1}{2}, 2)$.

$$f\left(\tfrac{1}{2}\right) = \tfrac{1}{2} + \tfrac{1}{\frac{1}{2}} = \tfrac{1}{2} + 2 = \tfrac{5}{2}$$

$$f(2) = 2 + \tfrac{1}{2} = \tfrac{5}{2}$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{\frac{5}{2} - \frac{5}{2}}{2 - \frac{1}{2}} = \frac{0}{2 - \frac{1}{2}} = 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

$\therefore$ By mean value theorem, $\exists\, c \in \left(\tfrac{1}{2}, 2\right) \ni f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\Rightarrow 1 - \frac{1}{c^2} = 0 \Rightarrow \frac{c^2 - 1}{c^2} = 0 \Rightarrow c^2 - 1 = 0 \Rightarrow c$$

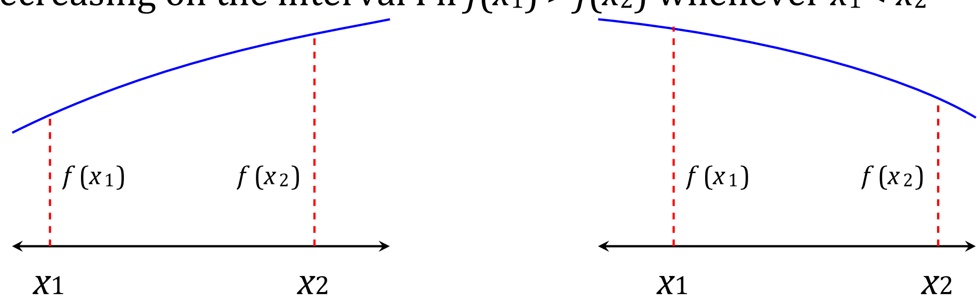
$$\therefore c = 1 \in \left(\tfrac{1}{2}, 2\right) \ni f'(1) = \frac{f(b) - f(a)}{b - a} = 0$$

$= 1$ and $c = -1$

Definition (23.2): Let f be defined on the interval I, and let x_1 and x_2 denote numbers in the interval I, then

1. f is increasing on the interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

2. f is decreasing on the interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$



Increasing Decreasing

Theorem (23.2):

Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b), then

1. If $f^{0}(x) > 0$ for every value of x in (a,b), then f is increasing function.

2. If $f^{0}(x) < 0$ for every value of x in (a,b), then f is decreasing function.

3. If $f^{0}(x) = 0$ for some x in (a,b), then x is critical point.

Example (23.3): Let $f(x) = 2x^2 + 4$

Solution:

$\because f^{0}(x) = 4x$

$\Rightarrow f^{0}(x) > 0 \ , \forall x > 0 \Rightarrow f$ increasing function.

$\Rightarrow f^{0}(x) < 0 \ , \forall x < 0 \Rightarrow f$ decreasing function.

$\Rightarrow f^{0}(x) = 0$ if $x = 0 \Rightarrow x$ is critical point.

Theorem (23.3):

Let f^{00} be twice differentiable on an open interval I, then

1. If $f^{00}(x) > 0$ on I, then f is concave up on I.

2. If $f^{00}(x) < 0$ on I, then f is concave down on I.

3. If $f^{00}(x) = 0$ for some x in I, then x is inflection point.

24. Curve Sketching With y^{0} and y^{00}

Steps of Graphing:

1. Find y^{0} and y^{00}.

2. Find y^0 is positive, negative and zero.

3. Find y^{00} is positive, negative and zero.

4. Make summary table.

5. Draw the graph.

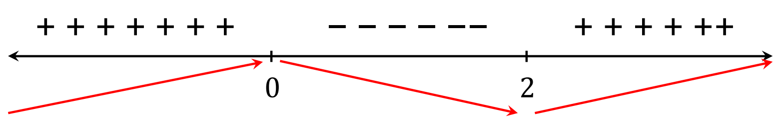
Example (24.1): Sketch the graph of $y = x^3 - 3x^2 + 4$

Solution:

$\Rightarrow y^0 = 3x^2 - 6x$

$\Rightarrow$ If $y^0 = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow x(3x - 6) = 0 \Rightarrow x = 0$ & $x = 2$

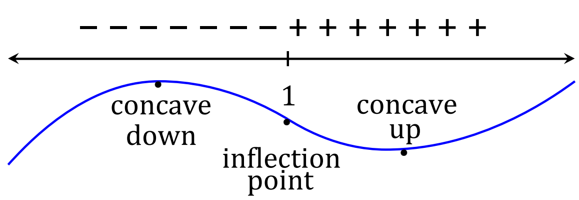
$\therefore$ (0,4) and (2,0) are critical points.



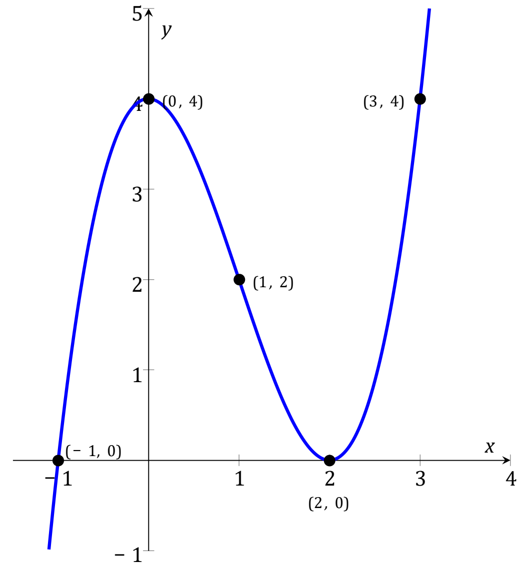
$\Rightarrow y^{00} = 6x - 6$

If $y^{00} = 0 \Rightarrow 6(x - 1) = 0 \Rightarrow x = 1$

$\therefore$ (1,2) is inflection point.



x	y	y0	y00	Behavior
-1	0	9	-12	concave down
0	4	0	-6	local maximum
1	2	-3	0	inflection point
2	0	0	6	local minimum
3	4	9	12	concave up



24.1 Asymptotes:

Definition (24.1): A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if one of the following conditions is true; $\lim_{x\to a} f(x) = \mp\infty$, $\lim_{x\to a^+} f(x) = \mp\infty$, $\lim_{x\to a^-} f(x) = \mp\infty$

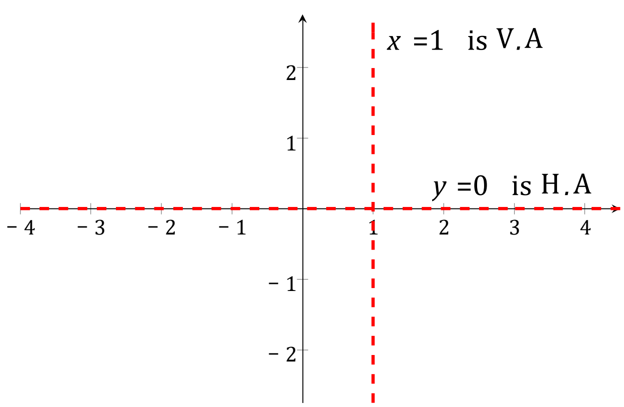
Example (24.2): Find the asymptotes of the curve $y = \frac{1}{x-1}$ **Solution:**

1) Horizontal asymptote

$$\Rightarrow \lim_{x\to\infty} \frac{1}{x-1} = 0 \Rightarrow y = 0 \text{ is } \mathcal{H.A}$$

2) Vertical asymptote

$$\Rightarrow \lim_{x\to 1} \frac{1}{x-1} = \infty \Rightarrow x = 1 \text{ is } \mathcal{V.A}$$



24.2 Oblique (Slant) Asymptotes:

If the function is $\frac{p(x)}{q(x)}$ such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $\frac{p(x)}{q(x)}$ will have an oblique asymptote by division of $p(x)$ by $q(x)$ to obtain

$$\frac{p(x)}{q(x)} = (ax+b) + \frac{r(x)}{q(x)}$$

Where $(ax + b)$ is the oblique asymptote.

Example (24.3): Find the oblique asymptote (O. A) for the function $y = \frac{x^2-3}{2x-4}$

Solution:

$$\begin{array}{r|l} & \frac{x}{2} + 1 \\ 2x-4 & x^2 - 3 \\ & \mp x^2 \pm 2x \\ & 2x - 3 \\ & \mp 2x \pm 4 \\ & 1 \end{array}$$

$$\therefore \frac{x^2-3}{2x-4} = \left(\frac{x}{2}+1\right) + \frac{1}{2x-4}$$

$\therefore y = \frac{x}{2} + 1$ is $\mathcal{O}.\ \mathcal{A}$

Example (24.4): Find the oblique asymptote (O. A) for the function $y = \frac{x^2+1}{x}$

Solution:

$$\begin{array}{r|l} & x \\ x & x^2 + 1 \\ & \mp x^2 \\ & 1 \end{array}$$

$$\therefore \frac{x^2+1}{x} = x + \frac{1}{x}$$

$\therefore y = x$ is O. A

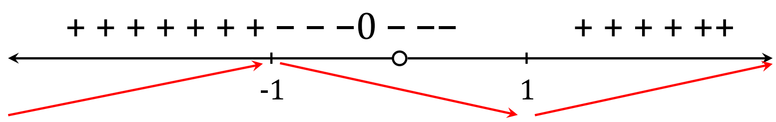
Example (24.5): Sketch the graph of $y = x + \frac{1}{x}$

Solution:

$\because y' = 1 - \frac{1}{x^2}$

$\Rightarrow y' = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \;\&\; x = -1$

$\therefore$ (1,2) and (−1,−2) are critical points.



$\because y'' = \frac{2}{x^3}$

$\Rightarrow y'' \neq 0 \Rightarrow$ there is no inflection point.

$\Rightarrow y^{00} > 0$ if $x > 0 \Rightarrow y$ is concave up.

$\Rightarrow y^{00} < 0$ if $x < 0 \Rightarrow y$ is concave down.

Asymptotes:

1. Horizontal asymptote

   $\lim_{x\to\infty} x + \frac{1}{x} = \infty \Rightarrow$ there is no horizontal asymptote

2. Vertical asymptote

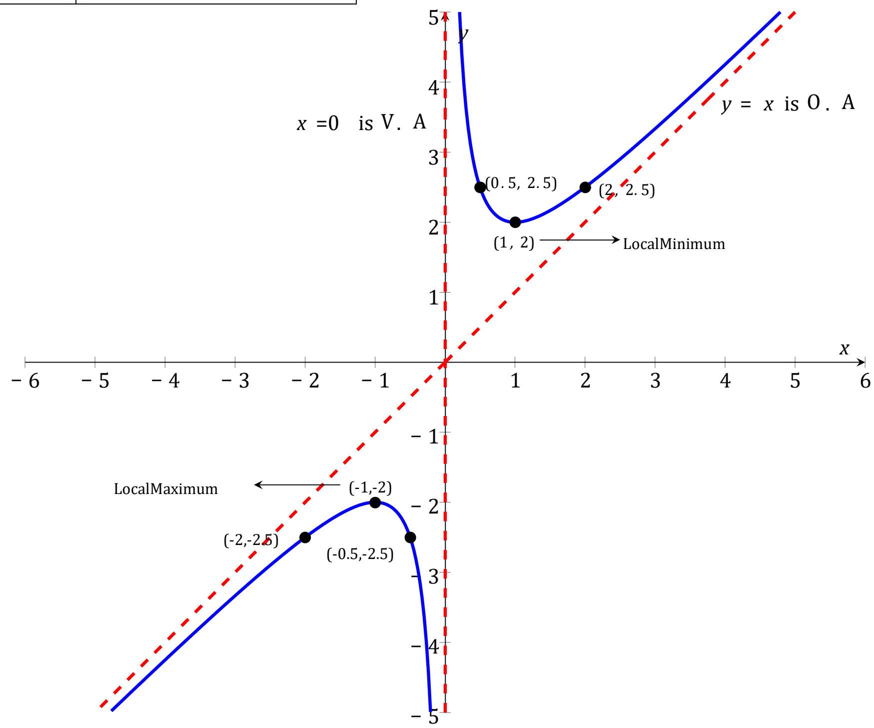
   $\because \lim_{x\to 0} x + \frac{1}{x} = \infty \Rightarrow x = 0$ is Vertical asymptote

3. Oblique asymptote

   $\therefore y = x$ is oblique asymptote

x	y	y_0	y_{00}	Behavior
-2	$\frac{-5}{2}$	$\frac{3}{4}$	$\frac{-1}{4}$	concave down
-1	-2	0	-2	local maximum

-0.5	-2.5	-3	-16	decreasing
0.5	2.5	-3	16	decreasing
1	2	0	2	local minimum
2	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	concave up



Exercises (24): Sketch a graph of the following functions (using y^0 and y^{00}):

1) $y = x^3 - 3x + 3$

2) $y = \frac{x^2}{x-1}$

3) $y = \frac{(x-1)^3}{x^2}$

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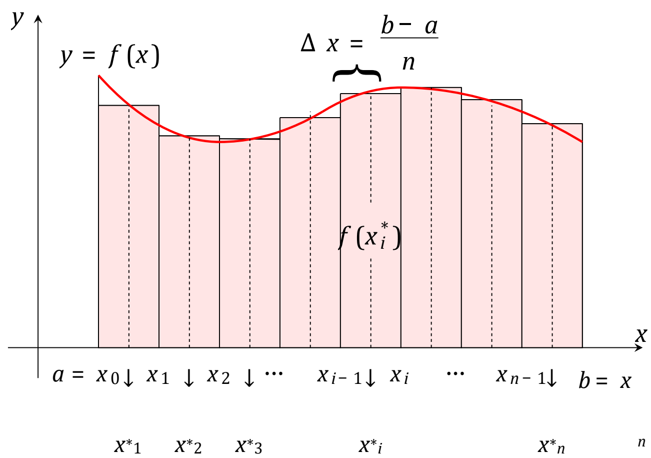
1

Integration

1.1 Definite Integral:



Given a function $f(x)$ that is continuous on the interval $[a,b]$, we divide the interval into "n"subinterval of equal width Δx, and from each interval choose point $x_i^* = a + (\Delta x)i$.

Then the definite integral of $f(x)$ from a to b is:

$$\int_a^b f(x)dx = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Properties of the definite Integral:

1) $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2) $$\int_a^a f(x)dx = 0$$

3) $\int_a^b k\, f(x)dx = k\int_a^b f(x)dx$, k is any number.

4) $\int_a^b (f(x) \mp g(x))dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

5) $\int_a^b f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$

6) $\int_a^b k\, dx = k(b-a)$, k is any number.

1

7) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

8) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

9) If $\alpha \leq f(x) \leq \beta$ for $a \leq x \leq b$, then $\alpha(b-a) \leq \int_a^b f(x)dx \leq \beta(b-a)$

10) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Remark (1.1):

1)

2)

3)

Example (1): Evaluate the integral $\int_0^3 (x^3 - 6x)dx$ by using definition.

Solution:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$a = 0\ ,\ \ b = 3,\ \ \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$\therefore x_1^* = \frac{3}{n},\ x_2^* = \frac{6}{n},\ x_3^* = \frac{9}{n},\ \cdots\ ,\ \text{in general}\quad x_i^* = \frac{3i}{n}$$

$$\therefore \int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left((x_i^*)^3 - 6\,x_i^*\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{81}{n^4}\,i^3 - \frac{54}{n^2}\,i\right) = \lim_{n\to\infty} \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left(\frac{n\,(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n\,(n+1)}{2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{81}{n^4}\left(\frac{n^4\left(1+\frac{1}{n}\right)^2}{4}\right)-\frac{54}{n^2}\left(\frac{n^2\left(1+\frac{1}{n}\right)}{2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{81}{4}\left(1+\frac{1}{n}\right)^2-\frac{54}{2}\left(1+\frac{1}{n}\right)\right)=\left(\frac{81}{4}\left(1+\frac{1}{\infty}\right)^2-\frac{54}{2}\left(1+\frac{1}{\infty}\right)\right)$$

$$= \frac{81}{4}-\frac{54}{2}=-\frac{27}{4}$$

1.2 Indefinite Integral:

Definition (2.1): A function $F(x)$ is called an *anti-derivative* of a function $f(x)$ if $F'(x)=f(x)$. If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an *indefinite integral* and denoted, $\int f(x)dx = F(x) + C$, C is any constant.

In this definition the $\int$ is called the *integral symbol*, $f(x)$ is called the *integrand*, x is called the *integration variable* and the "C" is called the *constant of integration*.

Properties of the Indefinite Integral:

1) $\int kf(x)dx = k\int f(x)dx$, k is any number

2) $\int -f(x)dx = -\int f(x)dx$

3) $\int (f(x) \mp g(x))dx = \int f(x)dx \mp \int g(x)dx$

4) $\int kdx = kx + C$, k and C are constant Remark (2.1):

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = u'(x)f(u(x))$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)dt = -v'(x)f(v(x))$$

1)

2)

3) $$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f(u(x)) - v'(x)f(v(x))$$

Example (1): Find the differentiate for each the following.

$$g(x) = \int_{-4}^{2x} e^{2t}\cos^2(1-5t)\,dt$$

$$g(x) = \int_{x^2}^{1} \frac{t^2+1}{t-1}dt$$

$$g(x) = \int_{\sin(x)}^{3x} t^2 \sin\left(1+t^2\right)dt$$

1)

2)

3)

Solution:

1) $$g'(x) = 2e^{4x}\cos^2(1-10x)$$

2) $$g'(x) = -2x\left(\frac{x^4+1}{x^2-1}\right)$$

3) $$g'(x) = 3\left(9x^2\sin\left(1+9x^2\right)\right) - \cos(x)\left(\sin^2(x)\sin\left(1+\sin^2(x)\right)\right)$$
$$= 27x^2\sin\left(1+9x^2\right) - \cos(x)\sin^2(x)\sin\left(1+\sin^2(x)\right)$$

Theorem (2.1):

1. $$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad , \ n \neq -1$$

2. $$\int (g(x))^n g'(x)dx = \frac{(g(x))^{n+1}}{n+1} + C \qquad , \ n \neq -1$$

Example (2): Evaluate each of the following integrals.

1) $$\int dx = x + C$$

2) $$\int 7dx = 7x + C$$

$$\int x^5 dx = \frac{x^6}{6} + C$$

$$\int x^{-3} dx = \frac{x^{-2}}{-2} + C$$

$$\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$\int \frac{1+x}{x^3} dx = \int \left(\frac{1}{x^3} + \frac{x}{x^3} \right) dx = \int \left(\frac{1}{x^3} + \frac{1}{x^2} \right) dx = \int \frac{1}{x^3} dx + \int \frac{1}{x^2} dx$$

$$= \int x^{-3} dx + \int x^{-2} dx = \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} + C = \frac{-1}{2x^2} - \frac{1}{x} + C$$

$$\int (x+1)^2 dx = \int (x^2 + 2x + 1) dx = \int x^2 dx + 2 \int x dx + \int dx$$

$$= \frac{x^3}{3} + \cancel{2} \frac{x^2}{\cancel{2}} + x + C$$

3)

4)

5)

6)

7)

Exercises (2.1): Evaluate each of the following integrals.

1)2)3)

$$\int x \left(1 + x^3\right) dx \qquad \int \left(2 + x^2\right)^2 dx \qquad \int x^{\frac{1}{3}} (2 - x)^2 dx$$

4)5)6)

$$\int \left(1 + x^2\right)(2 - x) dx \qquad \int \frac{1}{2x^3} dx \qquad \int \frac{x^5 + 2x^2 - 1}{x^4} dx$$

Example (3): Evaluate each of the following integrals.

1) $\int \frac{xdx}{\sqrt{(1-2x^2)^3}}$ 2) $\int \left(2x^3+1\right)^7 x^2 dx$ 3) $\int \left(x^2+3x+1\right)^5 (2x+3)\,dx$

Solution:

$$\int \frac{xdx}{\sqrt{(1-2x^2)^3}} = \int x\left(1-2x^2\right)^{-\frac{3}{2}} dx \times \frac{-4}{-4} = \frac{-1}{4}\int (-4x)\left(1-2x^2\right)^{-\frac{3}{2}} dx$$

$$= \frac{-1}{4}\left(\frac{\left(1-2x^2\right)^{-\frac{1}{2}}}{-\frac{1}{2}}\right) + C = \frac{1}{2\sqrt{1-2x^2}} + C$$

$$\int \left(2x^3+1\right)^7 x^2 dx = \int \left(2x^3+1\right)^7 x^2 dx \times \frac{6}{6} = \frac{1}{6}\int \left(2x^3+1\right)^7 \left(6x^2\right) dx$$

$$= \frac{1}{6}\left(\frac{\left(2x^3+1\right)^8}{8}\right) + C = \frac{1}{48}\left(2x^3+1\right)^8 + C$$

1)

2)

3) $\int \left(x^2+3x+1\right)^5 (2x+3)\,dx = \frac{\left(x^2+3x+1\right)^6}{6} + C$

Theorem (2.2):

Suppose $f(x)$ is continuous function on $[a,b]$ and also suppose that $F(x)$ is any antiderivative for $f(x)$, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Example (4): Evaluate each of the following integrals.

$$\int_{2}^{10} \frac{3dx}{\sqrt{5x-1}}$$

$$\int_{-1}^{3} \left(3x^2 - 2x + 1\right) dx$$

1)

2)

Solution:

$$\int_{2}^{10} \frac{3dx}{\sqrt{5x-1}} = \int_{2}^{10} 3\,(5x-1)^{-\frac{1}{2}}\,dx \times \frac{5}{5} = \frac{3}{5}\int_{2}^{10} 5\,(5x-1)^{-\frac{1}{2}}\,dx$$

$$= \frac{3}{5}\left[\frac{(5x-1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{2}^{10} = \frac{6}{5}\left(\sqrt{49} - \sqrt{9}\right) = \frac{6}{5}(7-3) = \frac{24}{5}$$

$$\int_{-1}^{3} \left(3x^2 - 2x + 1\right) dx = \int_{-1}^{3} 3x^2 dx - \int_{-1}^{3} 2x dx + \int_{-1}^{3} dx$$

$$= 3\int_{-1}^{3} x^2 dx - 2\int_{-1}^{3} x dx + \int_{-1}^{3} dx$$

$$= 3\left[\frac{x^3}{3}\right]_{-1}^{3} - 2\left[\frac{x^2}{2}\right]_{-1}^{3} + [x]_{-1}^{3}$$

$$= 3\left(\frac{27}{3} + \frac{1}{3}\right) - 2\left(\frac{9}{2} - \frac{1}{2}\right) + (3+1)$$

$$= \cancel{3}\left(\frac{28}{\cancel{3}}\right) - \cancel{2}\left(\frac{8}{\cancel{2}}\right) + 4 = 32 - 8 = 24$$

1)

2)

Exercises (2.2): Evaluate the following integrals.

1) $\int \frac{3x\,dx}{\sqrt{4x^2+5}}$ 2) $\int y\sqrt{1+2y^2}\,dy$ 3) $\int \frac{(1+x)^2}{\sqrt{x}}dx$ 4) $\int \frac{dx}{\sqrt{2x}\sqrt{5+\sqrt{x}}}$

1.3 Integration of The Trigonometric Functions:

1) $\int \sin(u)du = -\cos(u) + C$

2) $\int \cos(u)du = \sin(u) + C$

3) $\int \tan(u)du = \ln|\sec(u)| + C$

4) $\int \cot(u)du = \ln|\sin(u)| + C$

5) $\int \sec(u)du = \ln|\sec(u) + \tan(u)| + C$

6) $\int \csc(u)du = \ln|\csc(u) - \cot(u)| + C$

7) $\int \sec^2(u)du = \tan(u) + C$

8) $\int \csc^2(u)du = -\cot(u) + C$

9) $\int \sec(u)\tan(u)du = \sec(u) + C$

10) $\int \csc(u)\cot(u)du = -\csc(u) + C$

Example (1): Evaluate each of the following integrals.

1) $\int \sin(2x)dx$ 2) $\int x^2\sin(x^3)dx$ 3) $\int \frac{dx}{\cos^2(2x)}$

4) $\int \sin^2(x)\cos(x)dx$ 5) $\int \sqrt{2+\cos(x)}\sin(x)dx$

Solution:

1) $\int \sin(2x)dx = \int \sin(2x)dx \times \frac{2}{2} = \frac{1}{2}\int 2\sin(2x)dx = \frac{-1}{2}\cos(2x) + C$

$$\int x^2\sin(x^3)dx = \int x^2\sin(x^3)dx \times \frac{3}{3} = \frac{1}{3}\int \left(3x^2\right)\sin(x^3)dx$$

$$= \frac{-1}{3}\cos(x^3) + C$$

$$\int \frac{dx}{\cos^2(2x)} = \int \sec^2(2x)dx \times \frac{2}{2} = \frac{1}{2}\tan(2x) + C$$

$$\int \sin^2(x)\cos(x)dx = \frac{\sin^3(x)}{3} + C$$

$$\int \sqrt{2+\cos(x)}\sin(x)dx = \int \sqrt{2+\cos(x)}\sin(x)dx \times \frac{-1}{-1}$$

$$= -\int (2+\cos(x))^{\frac{1}{2}}\left(-\sin(x)\right)dx$$

$$= -\left(\frac{(2+\cos(x))^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = -\frac{2}{3}\left(2+\cos(x)\right)^{\frac{3}{2}} + C$$

2)

3)

4)

5)

Exercises (3.1): Evaluate each of the following integrals.

$\int (1+\tan(x))^2\,dx$ 2) $\int \frac{dx}{1+\cos(x)}$ 1) 3) $\int \frac{\sin(x)+\cos(x)}{\cos(x)}dx$ 4) $\int \frac{\cos(x)}{\sin^2(x)}dx$

$\int \left(\sqrt{\frac{\sin(x)}{x}} + \sqrt{\frac{x}{\sin(x)}}\cos(x)\right)dx$

5)

1.4 Integration of Exponential and Logarithmic Functions:

1) $\int \frac{du}{u} = \ln|u| + C$

2) $\int e^u du = e^u + C$

3) $\int a^u du = \frac{a^u}{\ln a} + C$

Example (1): Evaluate the following integrals.

1) $\int \frac{3x^2 dx}{x^3+5} = \ln|x^3+5| + C$

$$\int \frac{\sin(x)}{2+\cos(x)}dx = \int \frac{\sin(x)}{2+\cos(x)}dx \times \frac{-1}{-1} = -\int \frac{-\sin(x)}{2+\cos(x)}dx$$

$$= -\ln|2+\cos(x)| + C$$

$$\int e^{3x}dx = \int e^{3x}dx \times \frac{3}{3} = \frac{1}{3}\int 3e^{3x}dx = \frac{1}{3}e^{3x} + C$$

$$\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx = \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx \times \frac{2}{2} = \frac{1}{2}\ln|e^{2x}-e^{-2x}| + C$$

$$\int 10^{3x}dx = \int 10^{3x}dx \times \frac{3}{3} = \frac{1}{3}\frac{10^{3x}}{\ln(10)} + C$$

$$\int 3^x dx = \frac{3^x}{\ln(3)} + C$$

$$\int \frac{x+1}{x^2+2x+3}dx = \int \frac{x+1}{x^2+2x+3}dx \times \frac{2}{2} = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx$$

$$= \frac{1}{2}\ln|x^2+2x+3| + C$$

$$\int \frac{dx}{x\ln(x)} = \int \frac{\frac{1}{x}}{\ln(x)}dx = \ln|\ln(x)| + C$$

2)

3)

4)

5)

6)

7)

8)

Example (2): Prove that $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$ Proof:

$$\int \sec(x)dx = \int \sec(x) \times \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx$$

$$= \ln|\sec(x)+\tan(x)| + C$$

Exercises (4.1): Prove that.

1) $\int \tan(x)dx = \ln|\sec(x)| + C$
2) $\int \cot(x)dx = \ln|\sin(x)| + C$
3) $\int \csc(x)dx = \ln|\csc(x) - \cot(x)| + C$

Exercises (4.2): Evaluate

1) $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}dx$
2) $\int (\tan(2x)+\sec(2x))^2 dx$
3) $\int \frac{dx}{1-\sin(\frac{1}{2}x)}$
4) $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$
5) $\int e^x\sqrt{1+e^x}dx$
6) $\int \frac{dx}{\sqrt{2x-5}}$
7) $\int e^{3\cos(2x)}\sin(2x)dx$
8) $\int \sqrt{1-\sqrt{x}}dx$

Products of Sines and Cosines:

1. We begin with integral of the form; $\int \sin^m(x)\cos^n(x)dx$ where m and n are nonnegative integers (positive or zero), we can divide the work into three cases:

 i. If m is odd, we write m as $2k+1$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to obtain $\sin^m(x) = \sin^{2k+1}(x) = \left(\sin^2(x)\right)^k \sin(x) = \left(1-\cos^2(x)\right)^k \sin(x)$.

$\int$

Example (3): Evaluate the integral $\sin^3(x)\cos^4(x)dx$ Solution:

$$\begin{aligned}\int \sin^3(x)\cos^4(x)dx &= \int \sin(x)\sin^2(x)\cos^4(x)dx \\ &= \int \sin(x)\left(1-\cos^2(x)\right)\cos^4(x)dx \\ &= \int \sin(x)\cos^4(x) - \sin(x)\cos^6(x)dx \\ &= \int \sin(x)\cos^4(x)dx - \int \sin(x)\cos^6(x)dx \\ &= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C\end{aligned}$$

Z

Example (4): Evaluate the integral $\sin^3(x)\cos^2(x)dx$ (H.W) ii. If n is odd, we write

n as $2k + 1$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$

to obtain, $\cos^n(x) = \cos^{2k+1}(x) = \left(\cos^2(x)\right)^k \cos(x) = \left(1-\sin^2(x)\right)^k \cos(x)$.

Example (5): Evaluate the integral $\int \sin^4(2x)\cos^3(2x)dx$ Solution:

$$\begin{aligned}
\int \sin^4(2x)\cos^3(2x)dx &= \int \sin^4(2x)\cos^2(2x)\cos(2x)dx \\
&= \int \sin^4(2x)\left(1-\sin^2(2x)\right)\cos(2x)dx \\
&= \int \cos(2x)\sin^4(2x) - \cos(2x)\sin^6(2x)dx \\
&= \int \cos(2x)\sin^4(2x)dx - \int \cos(2x)\sin^6(2x)dx \\
&= \frac{\sin^5(2x)}{10} - \frac{\sin^7(2x)}{14} + C
\end{aligned}$$

Example (6): Evaluate the integral $\int \sin^2(x)\cos^5(x)dx$ (H.W)

iii. If both m and n are even, we substitute

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2} \quad ,$$

Example (7): Evaluate the integral $\int \cos^2(x)\sin^4(x)dx$

Solution:

$$\int \cos^2(x)\sin^4(x)dx = \int \cos^2(x)\left(\sin^2(x)\right)^2 dx$$
$$= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)^2 dx$$
$$= \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{(1-\cos(2x))^2}{4}\right) dx$$
$$= \frac{1}{8}\int (1+\cos(2x))\left(1-2\cos(2x)+\cos^2(2x)\right) dx$$
$$= \frac{1}{8}\int \left(1-2\cos(2x)+\cos^2(2x)+\cos(2x)-2\cos^2(2x)+\cos^3(2x)\right) dx$$
$$= \frac{1}{8}\int \left(1-\cos(2x)-\cos^2(2x)+\cos^3(2x)\right) dx$$
$$= \frac{1}{8}\left[\int dx - \int \cos(2x)dx - \int \cos^2(2x)dx + \int \cos^3(2x)dx\right]$$
$$\because \int \cos^2(2x)dx = \int \left(\frac{1+\cos(4x)}{2}\right) dx = \frac{1}{2}x + \frac{1}{8}\sin(4x) + C$$
$$\because \int \cos^3(2x)dx = \int \cos(2x)\left(1-\sin^2(2x)\right) dx$$
$$= \int \cos(2x) - \cos(2x)\sin^2(2x)dx$$
$$= \frac{1}{2}\sin(2x) - \frac{\sin^3(2x)}{6} + C$$
$$= \frac{1}{8}\left[x - \frac{1}{2}\sin(2x) - \frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{2}\sin(2x) - \frac{\sin^3(2x)}{6} + C\right]$$

Z Z

2. The integrals $\sin^m(x)dx$ or $\cos^m(x)dx$ where m is nonnegative integer.

—

i. If m is even we can use the identity;

$$\sin^2(x) = \left(\frac{1-\cos(2x)}{2}\right)$$

—

or

$$\cos^2(x) = \left(\frac{1+\cos(2x)}{2}\right)$$

Z Example

(8): Evaluate the integral $\cos^4(x)dx$ Solution:

—

$$\begin{aligned}
\int \cos^4(x)dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int \left(1+2\cos(2x)+\cos^2(2x)\right)dx \\
&= \frac{1}{4}\left[\int dx + 2\int \cos(2x)dx + \int \cos^2(2x)dx\right] \\
&= \frac{1}{4}\left[\int dx + \int 2\cos(2x)dx + \frac{1}{2}\int (1+\cos(4x))\,dx\right] \\
&= \frac{1}{4}\left[\int dx + \int 2\cos(2x)dx + \frac{1}{2}\left(\int dx + \int \cos(4x)dx\right)\right] \\
&= \frac{1}{4}\left[x + \sin(2x) + \frac{1}{2}\left(x + \frac{1}{4}\sin(4x)\right) + C\right] \\
&= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Example (9): Evaluate the following integrals:

1) $\int \cos^2(2x)dx$ 2) $\int \sin^4(2x)dx$ (H.W)

ii. If m is odd we write m as $2k+1$ and use the identity; $\sin^2(x) = 1 - \cos^2(x)$ <u>or</u> $\cos^2(x) = 1 - \sin^2(x)$

Example (10): Evaluate the integral $\int \cos^5(x)dx$ Solution:

$$\begin{aligned}
\int \cos^5(x)dx &= \int \cos^4(x)\cos(x)dx = \int \left(1-\sin^2(x)\right)^2\cos(x)dx \\
&= \int \left(1-2\sin^2(x)+\sin^4(x)\right)\cos(x)dx \\
&= \int \cos(x)dx - 2\int \sin^2(x)\cos(x)dx + \int \sin^4(x)\cos(x)dx \\
&= \sin(x) - \frac{2}{3}\sin^3(x) + \frac{\sin^5(x)}{5} + C
\end{aligned}$$

3. The integrals $\int \sin(mx)\sin(nx)dx$, $\int \sin(mx)\cos(nx)dx$, $\int \cos(mx)\cos(nx)dx$

$\int$

i. $\sin(mx)\sin(nx)dx$ (m and n are different). we use the identity;

$$\sin(mx)\sin(nx) = \frac{1}{2}\left[\cos(mx - nx) - \cos(mx + nx)\right]$$

Z

Example (11): Evaluate the integral $\sin(3x)\sin(2x)dx$ Solution:

$$\int \sin(3x)\sin(2x)dx = \int \frac{1}{2}\left[\cos(3x - 2x) - \cos(3x + 2x)x\right] dx$$

$$= \frac{1}{2}\int \cos(x)dx - \frac{1}{2}\int \cos(5x)dx$$

$$= \frac{1}{2}\sin(x) - \frac{1}{10}\sin(5x) + C$$

Z ii. $\sin(mx)\cos(nx)dx$ (m and n are different). we use the identity;

$$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin(mx + nx) + \sin(mx - nx)\right]$$

Z

Example (12): Evaluate the integral $\cos(5x)\sin(3x)dx$

Solution:

$$\int \cos(5x)\sin(3x)dx = \int \frac{1}{2}\left[\sin(8x) + \sin(-2x)\right] dx$$

$$= \frac{1}{2}\int \sin(8x)dx - \frac{1}{2}\int \sin(2x)dx = \frac{-1}{16}\cos(8x) + \frac{1}{4}\cos(2x) + C$$

Z iii. $\cos(mx)\cos(nx)dx$ (m and n are different). we use the identity;

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos(mx+nx)+\cos(mx-nx)\right]$$

Z

Example (13): Evaluate the integral $\cos(4x)\cos(2x)dx$ Solution:

$$\int \cos(4x)\cos(2x)dx = \int \frac{1}{2}\left[\cos(6x)+\cos(2x)\right]dx$$

$$= \frac{1}{2}\int \cos(6x)dx + \frac{1}{2}\int \cos(2x)dx = \frac{1}{12}\sin(6x)+\frac{1}{4}\sin(2x)+C$$

Z Example

(14): Evaluate the integral $\sec^4(2x)dx$ Solution:

$$\int \sec^4(2x)dx = \int \sec^2(2x)\sec^2(2x)dx = \int \left(\tan^2(2x)+1\right)\sec^2(2x)dx$$

$$= \int \tan^2(2x)\sec^2(2x)+\sec^2(2x)dx$$

$$= \int \tan^2(2x)\sec^2(2x)dx + \int \sec^2(2x)dx$$

$$= \frac{1}{6}\tan^3(2x)+\frac{1}{2}\tan(2x)+C$$

Z

Example (15): Evaluate the integral $\tan^4(x)dx$ (H.W)

Z

Example (16): Evaluate the integral $\tan^2(x)\sec^4(x)dx$

Solution:

$$\begin{aligned}\int \tan^2(x)\sec^4(x)dx &= \int \tan^2(x)\sec^2(x)\sec^2(x)dx \\ &= \int \tan^2(x)\left(\tan^2(x)+1\right)\sec^2(x)dx \\ &= \int \left(\tan^4(x)\sec^2(x)+\tan^2(x)\sec^2(x)\right)dx \\ &= \int \tan^4(x)\sec^2(x)dx + \int \tan^2(x)\sec^2(x)dx \\ &= \frac{\tan^5(x)}{5}+\frac{\tan^3(x)}{3}+C\end{aligned}$$

Z

Example (17): Evaluate the integral $\tan^3(x)\sec^3(x)dx$ Solution:

$$\begin{aligned}\int \tan^3(x)\sec^3(x)dx &= \int \tan^2(x)\sec^2(x)\sec(x)\tan(x)dx \\ &= \int \left(\sec^2(x)-1\right)\sec^2(x)\sec(x)\tan(x)dx \\ &= \int \sec^4(x)\sec(x)\tan(x)dx - \int \sec^2(x)\sec(x)\tan(x)dx \\ &= \frac{\sec^5(x)}{5}-\frac{\sec^3(x)}{3}+C\end{aligned}$$

Exercises (4.3): Evaluate the following integrals.

1) $\int_0^{2\pi} \sin^4(x)\cos^2(x)dx$ 2) $\int \frac{\cos^3(x)}{1-\sin(x)}dx$ 3) $\int_0^{\frac{\pi}{4}} \frac{\sin^2(\theta)}{\cos^2(\theta)}d\theta$ 4) $\int \frac{\cot^3(x)}{\csc(x)}dx$

5) $\int \cos(2y)\sin(\frac{1}{2}y)dy$ 6) $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos(x)}dx$ 7) $\int \tan^3(2\theta)\sec^3(2\theta)d\theta$

1.5 Integration of The Inverse Trigonometric Functions:

1)
$$\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \sin^{-1}\left(\frac{u}{a}\right) + C \\ -\cos^{-1}\left(\frac{u}{a}\right) + C \end{cases}$$

2)
$$\int \frac{du}{a^2+u^2} = \begin{cases} \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \\ -\frac{1}{a}\cot^{-1}\left(\frac{u}{a}\right) + C \end{cases}$$

3)
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \begin{cases} \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C \\ -\frac{1}{a}\csc^{-1}\left(\frac{u}{a}\right) + C \end{cases}$$

Example (1): Evaluate the integral $\int \frac{dx}{1+3x^2}$

Solution:

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = 3x^2 \Rightarrow u = \sqrt{3}x \Rightarrow du = \sqrt{3}dx$

$$\Rightarrow \int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}dx}{1+\left(\sqrt{3}x\right)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C$$

Example (2): Evaluate the integral $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

Solution:

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = e^{2x} \Rightarrow u = e^x \Rightarrow du = e^x dx$

$$\Rightarrow \int \frac{e^x dx}{\sqrt{1-(e^x)^2}} = \sin^{-1}(e^x) + C$$

Example (3): Evaluate the integral $\int \frac{dx}{x\sqrt{4x^2-9}}$

Solution:

$a^2 = 9 \Rightarrow a = 3$ and $u^2 = 4x^2 \Rightarrow u = 2x \Rightarrow du = 2dx$

$$\Rightarrow \int \frac{dx}{x\sqrt{4x^2-9}} \times \frac{2}{2} = \int \frac{2dx}{2x\sqrt{(2x)^2-(3)^2}} = \frac{1}{3}\sec^{-1}\left(\frac{2x}{3}\right) + C$$

Example (4): Evaluate the integral $\int \frac{\sec^2(x)dx}{\sqrt{1-\tan^2(x)}}$

Solution:

$$\int \frac{\sec^2(x)dx}{\sqrt{1-\tan^2(x)}} = \int \frac{\sec^2(x)dx}{\sqrt{1-(\tan(x))^2}}$$

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = (\tan(x))^2 \Rightarrow u = \tan(x) \Rightarrow du = \sec^2(x)dx$

$$\Rightarrow \int \frac{\sec^2(x)dx}{\sqrt{1-(\tan(x))^2}} = \sin^{-1}(\tan(x)) + C$$

Exercises (5.1): Evaluate the following integrals.

1) $\int \frac{dx}{\sqrt{1-4x^2}}$ 2) $\int \frac{dx}{1+16x^2}$ 3) $\int \frac{dx}{x\sqrt{x^2-1}}$

4) $\int \frac{e^{-x}dx}{\sqrt{1-e^{-2x}}}$ 5) $\int \frac{\sin(\theta)d\theta}{\cos^2(\theta)+1}$ 6) $\int \frac{dx}{x\sqrt{1-(\ln(x))^2}}$

7) $\displaystyle\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$ 8) $\displaystyle\int \frac{dx}{e^x + e^{-x}}$ 9) $\displaystyle\int \frac{\sin^{-1}(x)dx}{\sqrt{1-x^2}}$

1.6 Integration of The Hyperbolic Functions:

1) $\int \sinh(u)du = \cosh(u) + C$

2) $\int \cosh(u)du = \sinh(u) + C$

3) $\int \tanh(u)du = \ln|\cosh(u)| + C$

4) $\int \coth(u)du = \ln|\sinh(u)| + C$

5) $\int \text{sech}^2(u)du = \tanh(u) + C$

6) $\int \text{csch}^2(u)du = -\coth(u) + C$

7) $\int \text{sech}(u)\tanh(u)du = -\text{sech}(u) + C$

8) $\int \text{csch}(u)\coth(u)du = -\text{csch}(u) + C$

Example (1): Evaluate the following integrals.

$$\int \sinh^5(x)\cosh(x)dx = \frac{\sinh^6(x)}{6} + C$$

$$\int e^{2x}\operatorname{sech}^2(e^{2x})dx = \frac{1}{2}\tanh\left(e^{2x}\right) + C$$

$$\int \sqrt{\tanh(x)}\operatorname{sech}^2(x)dx = \frac{\tanh^{\frac{3}{2}}(x)}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{\tanh^3(x)} + C$$

$$\int \tanh(x)\operatorname{sech}^3(x)dx = -\int -\tanh(x)\operatorname{sech}(x)\operatorname{sech}^2(x)dx = \frac{-\operatorname{sech}^3(x)}{3} + C$$

$$\int \operatorname{sech}^2(2x-1)dx \times \frac{2}{2} = \frac{1}{2}\tanh(2x-1) + C$$

1)

2)

3)

4)

5)

$$\int e^x \sinh(x)dx = \int e^x\left(\frac{e^x - e^{-x}}{2}\right)dx = \int \frac{e^{2x}-1}{2}dx = \frac{1}{2}\int \left(e^{2x}-1\right)dx$$

$$\int e^x \sinh(x)dx = \frac{1}{2} \times \frac{1}{2}e^{2x} - \frac{1}{2}x + C = \frac{1}{4}e^{2x} - \frac{1}{2}x + C$$

6)

Exercises (6.1): Evaluate the following integrals.

Z ZZ $\int \cosh(\frac{x}{9})dx$

1) $\operatorname{sech}(x)dx$ 2) $\coth^2(3x)dx$3)4) $e^x\cosh(x)dx$

1.7 Integration of The Inverse Hyperbolic Functions:

1) $$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

2) $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$$

3) $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } |u| < a \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C & \text{if } |u| > a \end{cases}$$

4) $$\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{-1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C$$

5) $$\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\operatorname{csch}^{-1}\left(\frac{u}{a}\right) + C$$

Example (1): Evaluate the integral $\int \frac{dx}{\sqrt{4x^2-9}}$

Solution:

$a^2 = 9 \Rightarrow a = 3$ and $u^2 = 4x^2 \Rightarrow u = 2x \Rightarrow du = 2dx$

$$\Rightarrow \int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2dx}{\sqrt{(2x)^2-(3)^2}} = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

Example (2): Evaluate the integral $\int \frac{dx}{\sqrt{1+9x^2}}$

Solution:

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = 9x^2 \Rightarrow u = 3x \Rightarrow du = 3dx$

$$\Rightarrow \int \frac{dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{3dx}{\sqrt{1+(3x)^2}} = \frac{1}{3}\sinh^{-1}(3x) + C$$

Example (3): Evaluate the integral $\int \frac{\tan(x)}{\sqrt{\sin^4(x)+\cos^4(x)}}\,dx$ Solution:

$$\int \frac{\tan(x)}{\sqrt{\sin^4(x)+\cos^4(x)}}\,dx = \int \frac{\tan(x)}{\sqrt{\cos^4(x)\left(\tan^4(x)+1\right)}}\,dx = \int \frac{\tan(x)}{\cos^2(x)\sqrt{\tan^4(x)+1}}\,dx$$

$$= \int \frac{\tan(x)\ \sec^2(x)}{\sqrt{\tan^4(x)+1}}\,dx = \int \frac{\tan(x)\ \sec^2(x)}{\sqrt{(\tan^2(x))^2+1}}\,dx$$

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = (\tan^2(x))^2 \Rightarrow u = \tan^2(x) \Rightarrow du = 2\tan(x)\sec^2(x)dx$

$$\Rightarrow \int \frac{\tan(x)\ \sec^2(x)}{\sqrt{(\tan^2(x))^2+1}}\,dx = \frac{1}{2}\sinh^{-1}(\tan^2(x)) + C$$

Example (4): Evaluate the integral $\int \frac{dx}{x\sqrt{1+4x^2}}$

Solution:

$a^2 = 1 \Rightarrow a = 1$ and $u^2 = 4x^2 \Rightarrow u = 2x \Rightarrow du = 2dx$

$$\Rightarrow \int \frac{dx}{x\sqrt{1+4x^2}} = \int \frac{2dx}{2x\sqrt{1+(2x)^2}} = -\operatorname{csch}^{-1}(2x) + C$$

Exercises (7.1): Evaluate the following integrals.

1) $\int \frac{dt}{\sqrt{t^2+1}}$ 2) $\int \frac{dx}{\sqrt{9x^2-25}}$ 3) $\int \frac{dx}{9x^2+25}$

1.8 The Methods of Integration:

1.8.1 Integration by Substitution:

Example (1): Evaluate the integral $\int 2x\sqrt{1+x^2}\,dx$

Solution:

let $u = 1 + x^2 \Rightarrow du = 2xdx$

$$\Rightarrow \int 2x\sqrt{1+x^2}dx = \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

Example (2): Evaluate the integral $\int \frac{2^x dx}{1+4^x}$

Solution:

let $u = 2^x \Rightarrow du = 2^x \ln(2)dx \Rightarrow 2^x dx = \frac{du}{\ln(2)}$

$$\Rightarrow \int \frac{2^x dx}{1+4^x} = \int \frac{2^x dx}{1+(2^2)^x} = \int \frac{2^x dx}{1+2^{2x}} = \int \frac{2^x dx}{1+(2^x)^2} = \int \frac{\frac{du}{\ln(2)}}{1+u^2}$$

$$= \frac{1}{\ln(2)}\int \frac{du}{1+u^2} = \frac{1}{\ln(2)}\tan^{-1}(u) + C = \frac{1}{\ln(2)}\tan^{-1}(2^x) + C$$

Example (3): Evaluate the integral $\int_0^{\frac{\pi}{4}} \tan(x)\sec^2(x)dx$ Solution: let $u = \tan(x) \Rightarrow du = \sec^2(x)dx$

$$\Rightarrow \text{If } x = \tfrac{\pi}{4} \Rightarrow u = \tan\left(\tfrac{\pi}{4}\right) = 1$$

$$\Rightarrow \text{If } x = 0 \Rightarrow u = \tan(0) = 0$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \tan(x)\sec^2(x)dx = \int_0^1 udu = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

Exercises (8.1.1): Evaluate the following integrals.

1) $\int \frac{2z}{\sqrt[3]{z^2+1}}dz$ 2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(x)}{(2+\sin(x))^2}dx$ 3) $\int \frac{dx}{\sqrt{1+\sqrt{x}}}$ 4) $\int \frac{\cos^3(x)+\cos^5(x)}{\sin^2(x)+\sin^4(x)}$

1.8.2 Integration by Completing the Square:

Example (4): Evaluate the integral $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x)}} = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}}$$

$$= \int \frac{dx}{\sqrt{-(x^2-2x+1)+1}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$$

Example (5): Evaluate the integral $\int \frac{dx}{4x^2+4x+2}$ Solution:

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{4\left(x^2+x+\frac{1}{2}\right)} = \int \frac{dx}{4\left(x^2+x+\frac{1}{4}+\frac{1}{4}\right)}$$

$$= \int \frac{dx}{4\left(x^2+x+\frac{1}{4}\right)+1} = \int \frac{dx}{4\left(x+\frac{1}{2}\right)^2+1}$$

$$= \int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\tan^{-1}(2x+1) + C$$

Example (6): Evaluate the integral $\int \frac{dx}{x^2+2x+2}$ Solution:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{x^2+2x+1+1} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1) + C$$

Exercises (8.2.1): Evaluate the following integrals.

1) $\int \frac{dx}{x^2+10x+30}$ 2) $\int \frac{dx}{\sqrt{20+8x-x^2}}$ 3) $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

1.8.3 Reducing an Improper Fraction:

Example (7): Evaluate the integral $\int \frac{x+1}{x+2}dx$

Solution:

$$\int \frac{x+1}{x+2}dx = \int \left(1 - \frac{1}{x+2}\right)dx$$

$$= x - \ln|x+2| + C$$

1

$x + 2$ z$x + 1$ ∓zx ∓ 2 −1

Example (8): Evaluate the integral $\int \frac{(x-2)^3}{x^2-4}dx$

Solution:

$\frac{(\ \ -2)}{x^2-4} = \frac{(\ \ -2)(\ \ -2)}{} = \frac{(\ \ -2)}{(x+2)}$

$(x-2)(x+2)$

$$x+2\ \overline{)\,x^2-4x+4}$$

$\mp x^2 \mp 2x$

$-6x+4$

$\pm 6x \pm 12$

16

$x-6$

$$\frac{(x-2)^3}{x^2-4} = \frac{x^2-4x+4}{x+2} = (x-6)+\frac{16}{x+2}$$

$$\therefore \int \frac{(x-2)^3}{x^2-4}dx = \int \left((x-6)+\frac{16}{x+2}\right)dx$$

$$\int \frac{(x-2)^3}{x^2-4}dx = \frac{x^2}{2} - 6x + 16\ln|x+2| + C$$

Example (9): Evaluate the integral $\int \frac{3x^3-4x^2+3x}{x^2+1}dx$

Solution:

$3x-4$

$$x^2+1\ \overline{)\,3x^3-4x^2+3x}$$

$\mp 3x^3 \mp 3x$

$-4x^2$

$\pm 4x^2 \pm 4$

4

$$\int \frac{3x^3-4x^2+3x}{x^2+1}dx = \int \left(3x-4+\frac{4}{x^2+1}\right)dx$$

$$\int \frac{3x^3-4x^2+3x}{x^2+1}dx = 3\int x\,dx - 4\int dx + 4\int \frac{dx}{x^2+1}$$

$$\int \frac{3x^3-4x^2+3x}{x^2+1}dx = \frac{3}{2}x^2 - 4x + 4\tan^{-1}(x) + C$$

1.8.4 Integration by Separating a Fraction

Example (10): Evaluate the integral $\int \frac{3x+2}{\sqrt{1-x^2}}dx$ Solution:

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx = 3\int \frac{x}{\sqrt{1-x^2}}dx + 2\int \frac{dx}{\sqrt{1-x^2}}$$

The first integral:

let $u = 1-x^2 \Rightarrow du = -2x\,dx \Rightarrow \frac{-1}{2}du = x\,dx$

$$\Rightarrow 3\int \frac{x}{\sqrt{1-x^2}}dx = 3\int \frac{\left(\frac{-1}{2}\right)}{\sqrt{u}}du = \frac{-3}{2}\int \frac{du}{\sqrt{u}} = \frac{-3}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C_1$$

$$= -3\sqrt{u} + C_1 = -3\sqrt{1-x^2} + C_1$$

The second integral:

$$\Rightarrow 2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}(x) + C_2$$

$$\therefore \int \frac{3x+2}{\sqrt{1-x^2}}dx = -3\sqrt{1-x^2} + 2\sin^{-1}(x) + C$$

Example (11): Evaluate the integral $\int_0^{\frac{\pi}{4}} \frac{1+\sin(x)}{\cos^2(x)}dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{1+\sin(x)}{\cos^2(x)}dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)dx = \int_0^{\frac{\pi}{4}} \sec^2(x)dx + \int_0^{\frac{\pi}{4}} \sin(x)\cos^{-2}(x)dx$$

$$= [\tan(x)]_0^{\frac{\pi}{4}} + \left[\cos^{-1}(x)\right]_0^{\frac{\pi}{4}} = [\tan(x)]_0^{\frac{\pi}{4}} + [\sec(x)]_0^{\frac{\pi}{4}} = \sqrt{2}$$

Example (12): Evaluate the integral $\int_0^{\frac{\sqrt{3}}{2}} \frac{1-x}{\sqrt{1-x^2}}dx$ (H.W)

1.8.5 Integration by Parts

The formula for integration by parts comes from the product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow udv = d(uv) - vdu$$

$$\Rightarrow \int udv = uv - \int vdu$$

يجب فصل التكامل المعطى الى جزئين احدهما u والاخر dv بحيث نختار u وتكون قابلة للتفاضل و dv
قابلة للتكامل ويجب ان لايكون التكامل $\int vdu$ اكثر تعقيداً من التكامل $\int udv$

The equivalent formula for definite integrals is:

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

Z Example

(13): Evaluate the integral $x\cos(x)dx$

Solution:

let $u = x \Rightarrow du = dx$

$dv = \cos(x)dx \Rightarrow v = \sin(x)$

$\therefore \int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + C$

Example (14): Evaluate the integral $\int \ln(x)dx$

Solution:

let $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

$\Rightarrow dv = dx \Rightarrow v = x$

$$\therefore \int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x + C$$

Example (15): Evaluate the integral $\int x^2e^xdx$

Solution:

let $u = x^2 \Rightarrow du = 2xdx$

$\Rightarrow dv = e^xdx \Rightarrow v = e^x$

$\therefore \int x^2e^xdx = x^2e^x - 2\int xe^xdx$ let $u = x \Rightarrow$

$du = dx$

$\Rightarrow dv = e^xdx \Rightarrow v = e^x$

$$\therefore \int x^2e^xdx = x^2e^x - 2\left[xe^x - \int e^xdx\right] = x^2e^x - 2xe^x + 2e^x + C$$

Example (16): Evaluate the integral $\int e^x\cos(x)dx$

Solution:

let $u = e^x \Rightarrow du = e^x dx$

$\Rightarrow dv = \cos(x)dx \Rightarrow v = \sin(x)$

$\therefore \int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx$ let $u =$

$e^x \Rightarrow du = e^x dx$

$$\Rightarrow dv = \sin(x)dx \Rightarrow v = -\cos(x)$$

$$\therefore \int e^x \cos(x)dx = e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)dx\right]$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx$$

$$\Rightarrow 2\int e^x \cos(x)dx = e^x \sin(x) + e^x \cos(x)$$

$$\Rightarrow \int e^x \cos(x)dx = \frac{1}{2}e^x \sin(x) + \frac{1}{2}e^x \cos(x) + C$$

Example (17): Evaluate the integral $\int \sin^{-1}(x)dx$

Solution:

let $u = \sin^{-1}(x) \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$

$$\Rightarrow dv = dx \Rightarrow v = x$$

$$\Rightarrow \int \sin^{-1}(x)dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1}(x) - \int x(1-x^2)^{\frac{-1}{2}}dx \times \frac{-2}{-2}$$

$$= x\sin^{-1}(x) + \frac{1}{2}\int (-2x)(1-x^2)^{\frac{-1}{2}}dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

Exercises (8.5.1): Evaluate each of the following integrals.

1) $\int x\sin(x)dx$ 2) $\int \sin(\ln(x))dx$ 3) $\int \tan^{-1}(x)dx$

4) $\int x^3 e^x dx$ 5) $\int x\ln(x)dx$ 6) $\int \ln(x^2+2)dx$ 7) $\int x\sec^{-1}(x)dx$

1.8.6 Tabular Integration:

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero, and g can be integrated repeatedly with out difficulty, are natural candidates for integration by parts.

Example (18): Evaluate the integral $\int x^2 e^x dx$ by tabular integration.

Solution:

$f(x) = x^2$, $g(x) = e^x$

$f(x)$ and its derivative		$g(x)$ and its integral
x^2		e^x
$2x$	$+$	e^x
2	$-$	e^x
0	$+$	e^x

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

Example (19): Evaluate the integral $\int x^3 \sin(x)dx$ by tabular integration.

Solution:

$f(x) = x^3$, $g(x) = \sin(x)$

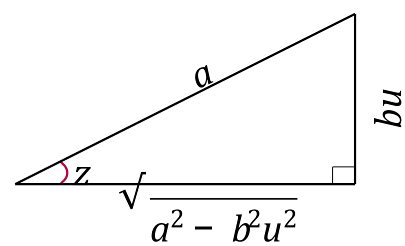
$f(x)$ and its derivative		$g(x)$ and its integral
x^3	+	$\sin(x)$
$3x^2$	−	$-\cos(x)$
$6x$	+	$-\sin(x)$
6	−	$\cos(x)$
0		$\sin(x)$

$$\Rightarrow \int x^3 \sin(x)dx = -x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x) + C$$

1.8.7 Trigonometric Substitutions:

i. The function of the form $\sqrt{a^2 - b^2u^2}$ we use the following

$$u = \frac{a}{b}\sin(z) \Rightarrow du = \frac{a}{b}\cos(z)dz$$
$$\Rightarrow \sin(z) = \frac{bu}{a} \Rightarrow z = \sin^{-1}\left(\frac{bu}{a}\right)$$ substitute

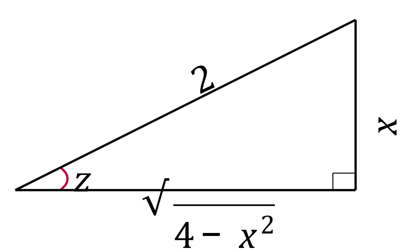


Example (20): Evaluate the integral $\int \frac{x^2}{\sqrt{4-x^2}}dx$ Solution:

$a = 2$, $b = 1$

let

$$\because \cos(z) = \frac{\sqrt{4-x^2}}{2} \Rightarrow \sqrt{4-x^2} = 2\cos(z)$$



$$x = 2\sin(z) \Rightarrow dx = 2\cos(z)dz \Rightarrow \sin(z) = \frac{x}{2}$$

$$\Rightarrow \int \frac{x^2}{\sqrt{4-x^2}}dx = \int \frac{4\sin^2(z)}{\cancel{2\cos(z)}}\cancel{2\cos(z)}dz = \int 4\sin^2(z)dz = 4\int \sin^2(z)dz$$

$$= 4\int \left(\frac{1-\cos(2z)}{2}\right)dz = 2\left(\int dz - \int \cos(2z)dz\right)$$

$$= 2z - \sin(2z) + C = 2z - 2\sin(z)\cos(z) + C$$

$$\because \sin(z) = \frac{x}{2} \Rightarrow z = \sin^{-1}\left(\frac{x}{2}\right)$$

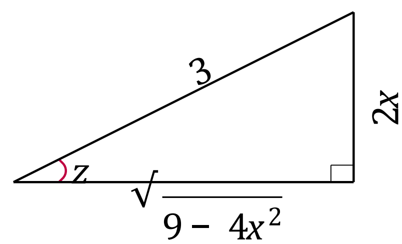
$$\Rightarrow \int \frac{x^2}{\sqrt{4-x^2}}dx = 2\sin^{-1}\left(\frac{x}{2}\right) - \cancel{2}\left(\frac{x}{\cancel{2}}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \left(\frac{x\sqrt{4-x^2}}{2}\right) + C$$

Example (21): Evaluate the integral $\int \frac{\sqrt{9-4x^2}}{x}dx$

Solution:

$a = 3$, $b = 2$



$$\because \cos(z) = \frac{\sqrt{9-4x^2}}{3} \Rightarrow \sqrt{9-4x^2} = 3\cos(z)$$

let $x = \frac{3}{2}\sin(z) \Rightarrow dx = \frac{3}{2}\cos(z)dz \Rightarrow \sin(z) = \frac{2x}{3}$

$$\Rightarrow \int \frac{\sqrt{9-4x^2}}{x}dx = \int \frac{3\cos(z)}{\frac{3}{2}\sin(z)} \times \frac{3}{2}\cos(z)dz = \int \frac{3\cos^2(z)}{\sin(z)}dz = 3\int \frac{1-\sin^2(z)}{\sin(z)}dz$$

$$= 3\left(\int \frac{dz}{\sin(z)} - \int \frac{\sin^2(z)}{\sin(z)}dz\right) = 3\left(\int \csc(z)dz - \int \sin(z)dz\right)$$

$$= 3\ln|\csc(z)-\cot(z)| + 3\cos(z) + C$$

$$= 3\ln\left|\frac{1}{\sin(z)} - \frac{\cos(z)}{\sin(z)}\right| + 3\cos(z) + C$$

$$= 3\ln\left|\frac{1}{2x/3} - \frac{\sqrt{9-4x^2}/3}{2x/3}\right| + 3\frac{\sqrt{9-4x^2}}{3} + C$$

$$= 3\ln\left|\frac{3}{2x} - \frac{\sqrt{9-4x^2}}{2x}\right| + \sqrt{9-4x^2} + C$$

? sin(HzH)

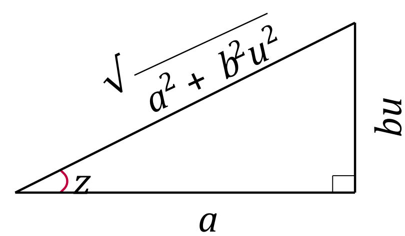
p 2 + b^2u^2 ii.

The function of the form a

we use the following

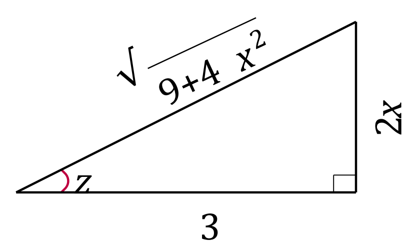
$$u = \frac{a}{b}\tan(z) \Rightarrow du = \frac{a}{b}\sec^2(z)dz$$

$$\Rightarrow \tan(z) = \frac{bu}{a} \Rightarrow z = \tan^{-1}\left(\frac{bu}{a}\right)$$ substitute



Example (22): Evaluate the integral $\int \frac{dx}{x\sqrt{9+4x^2}}$ Solution:

$a = 3$, $b = 2$



let $x = \frac{3}{2}\tan(z) \Rightarrow dx = \frac{3}{2}\sec^2(z)dz \Rightarrow \tan(z) = \frac{2x}{3}$

$$\because \cos(z) = \frac{3}{\sqrt{9+4x^2}} \Rightarrow \sec(z) = \frac{\sqrt{9+4x^2}}{3}$$

$$\Rightarrow \sqrt{9+4x^2} = 3\sec(z)$$

$$\because \sin(z) = \frac{2x}{\sqrt{9+4x^2}}$$

$$\Rightarrow \int \frac{dx}{x\sqrt{9+4x^2}} = \int \frac{\frac{3}{2}\sec^2(z)\,dz}{\frac{3}{2}\tan(z) * 3\sec(z)} = \frac{1}{3}\int \frac{\sec(z)}{\tan(z)}dz$$

$$= \frac{1}{3}\int \csc(z)dz = \frac{1}{3}\ln|\csc(z) - \cot(z)| + C$$

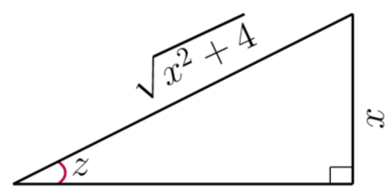
$$= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}}{2x} - \frac{3/\sqrt{9+4x^2}}{2x/\sqrt{9+4x^2}}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}}{2x} - \frac{3}{2x}\right| + C = \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right| + C$$

Example (23): Evaluate the integral $\int \frac{dx}{\sqrt{x^2+4}}$ Solution:

$a = 2\ ,\ \ b = 1$

$$x = 2\tan(z) \Rightarrow dx = 2\sec^2(z)dz \Rightarrow \tan(z) = \frac{x}{2}$$



$$\because \cos(z) = \frac{2}{\sqrt{x^2+4}} \Rightarrow \sqrt{x^2+4} = 2\sec(z)$$

$$\because \sin(z) = \frac{x}{\sqrt{x^2+4}}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2(z)\,dz}{2\sec(z)} = \int \sec(z)dz = \ln|\sec(z) + \tan(z)| + C$$

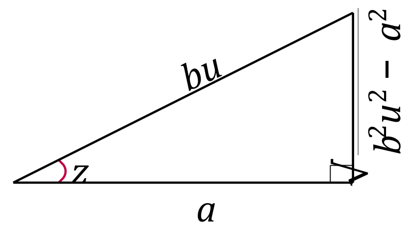
$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x/\sqrt{x^2+4}}{2/\sqrt{x^2+4}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C$$

let

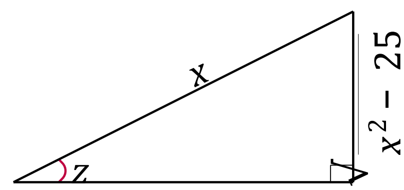
iii. The function of the form $\sqrt{b^2u^2 - a^2}$ we use the following substitute

$$u = \frac{a}{b}\sec(z) \Rightarrow du = \frac{a}{b}\sec(z)\tan(z)dz$$
$$\Rightarrow \sec(z) = \frac{bu}{a} \Rightarrow z = \sec^{-1}\left(\frac{bu}{a}\right)$$



Example (24): Evaluate the integral $\int \frac{dx}{\sqrt{x^2 - 25}}$

Solution:



$a = 5$, $b = 1$

5 let $x = 5\sec(z) \Rightarrow dx = 5\sec(z)\tan(z)dz$

$\Rightarrow x^2 - 25 = 25\sec^2(z) - 25 = 25(\sec^2(z) - 1) = 25\tan^2(z)$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2 - 25}} = \int \frac{5\sec(z)\tan(z)dz}{\sqrt{25\tan^2(z)}} = \int \frac{5\sec(z)\tan(z)}{5\tan(z)}dz = \int \sec(z)dz$$

$$= \ln|\sec(z) + \tan(z)| + C = \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}/x}{5/x}\right| + C$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C = \ln\left|\frac{x + \sqrt{x^2-25}}{5}\right| + C$$

Exercises (8.7.1): Evaluate the following integrals.

1) $\int \frac{\sqrt{x^2 - 25}}{x}dx$ 2) $\int \frac{x^2}{\sqrt{5 + x^2}}dx$

1.8.8 Integration by Partial Fractions:

The method of partial fractions is used to integrate rational functions $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and the degree of $P(x)$ *is less than* the degree of

$Q(x)$. If the degree of numerator greater than or equal the degree of the denominator, then must use long division firstly.

Remake (8.8.1):

1. First factor the denominator terms with simpler form.

2. If there is a factor has a fraction degree, then can not use this method to solve the given integral.

There are four cases to partial fractions:

Case 1: The denominator has only first degree factors, none of which are repeated.

Example (25): Evaluate the integral $\int \frac{dx}{x^2-4}$

Solution:

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \qquad \times (x-2)(x+2)$$

$$\Rightarrow 1 = A(x+2) + B(x-2) \Rightarrow 1 = Ax + 2A + Bx - 2B \Rightarrow (A+B)x + (2A-2B) = 1$$

$$A + B = 0 \cdots (1) \qquad \times 2 \Rightarrow \qquad 2A + 2B = 0 \cdots (1)$$

$$\underline{2A - 2B = 1} \cdots (2) \Rightarrow \qquad \underline{\mp 2A \pm 2B = \mp 1} \cdots (2)$$

$$4B = -1 \Rightarrow \qquad \boxed{B = \frac{-1}{4}} \Rightarrow \boxed{A = \frac{1}{4}}$$

$$\Rightarrow \int \frac{dx}{x^2-4} = \int \frac{\frac{1}{4}}{x-2}\,dx - \int \frac{\frac{1}{4}}{x+2}\,dx = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2}$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

Example (26): Evaluate the integral $\int \frac{5x-3}{x^2-2x-3}dx$

Solution:

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x-3)(x+1)} = \frac{A}{x+1} + \frac{B}{x-3} \qquad \times (x-3)(x+1)$$

$$\Rightarrow 5x-3 = A(x-3) + B(x+1) \Rightarrow 5x-3 = Ax - 3A + Bx + B$$

$$\Rightarrow 5x-3 = (A+B)x + (B-3A)$$

$$A + B = 5 \cdots (1) \qquad \times 3$$

$$-3A + B = -3 \cdots (2)$$

$$4B = 12 \Rightarrow \boxed{B = 3} \Rightarrow \boxed{A = 2}$$

$$\Rightarrow \frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

$$\Rightarrow \int \frac{5x-3}{x^2-2x-3}dx = 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3} = 2\ln|x+1| + 3\ln|x-3| + C$$

Exercises (8.8.1): Evaluate the following integrals.

1) $\int \frac{5x-10}{x^2-3x-4}dx$ 2) $\int \frac{x+1}{x^3+x^2-6x}dx$

Case 2: The denominator has only first degree factors, but some of these factors may be repeated factors.

Example (27): Evaluate the integral $\int \frac{3x+5}{x^3-x^2-x+1}dx$ Solution:

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x^3-x)-(x^2-1)} = \frac{3x+5}{x(x^2-1)-(x^2-1)} = \frac{3x+5}{(x^2-1)(x-1)}$$

$$= \frac{3x+5}{(x-1)(x+1)(x-1)} = \frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{D}{(x-1)^2}$$

$$\Rightarrow 3x+5 = A(x-1)^2 + B(x+1)(x-1) + D(x+1)$$

$$\Rightarrow 3x+5 = A(x^2-2x+1) + B(x^2-1) + D(x+1)$$

$\Rightarrow 3x + 5 = Ax^2 - 2Ax + A + Bx^2 - B + Dx + D$

$\Rightarrow 3x + 5 = (A + B)x^2 + (D - 2A)x + (A - B + D)$

$A + B = 0 \cdots (1)$

$-2A + D = 3 \cdots (2)$

$A + D - B = 5 \cdots (3)$

from (1) and (3) $\Rightarrow 2A + D = 5 \cdots (4)$

$\underline{-2A + D = 3} \cdots (2)$

$2D = 8 \Rightarrow \boxed{D = 4}$

$-2A + 4 = 3 \Rightarrow \boxed{A = \frac{1}{2}} \Rightarrow \boxed{B = \frac{-1}{2}}$

$$\Rightarrow \int \frac{3x+5}{x^3 - x^2 - x + 1}dx = \int \frac{\frac{1}{2}}{x+1}dx + \int \frac{4}{(x-1)^2}dx - \int \frac{\frac{1}{2}}{x-1}dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C$$

Example (28): Evaluate the integral $\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}dx$

Solution:

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2}$$

$$\frac{x+1}{x^3 - x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{D}{x-1}$$

$$\Rightarrow x + 1 = Ax(x-1) + B(x-1) + Dx^2$$

$\Rightarrow x + 1 = Ax^2 - Ax + Bx - B + Dx^2$

$\Rightarrow x + 1 = (A + D)x^2 + (-A + B)x - B$

$\Rightarrow A + D = 0 \cdots (1)$

$-A + B = 1 \cdots (2)$

$-B = 1 \Rightarrow \boxed{B = -1} \Rightarrow \boxed{A = -2} \Rightarrow \boxed{D = 2}$

$\Rightarrow$ Z $x^4 - x^3 - x - 1$ ZZ −2 Z

______dx =

xdx − ___dx − $x^3 - x^2$

x

x^2 1

= ___ + 2ln|x| −

_ − 2ln 2

x

x $\underline{x^3}$ $\underline{- x^2}$ @x@^{4} −

@x@$^3 - x - 1$

$$\mp x^4 \pm x^3$$

$$-x-1$$

$$\frac{-dx}{x^2} - \int \frac{2dx}{x-1}$$

$|x - 1| + C$

Case 3: The denominator has one or more quadratic factors, none of which are repeated.

Example (29): Evaluate the integral $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$

Solution:

$$\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{x^3+x^2+x+2}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Dx+F}{x^2+1}$$

$$\Rightarrow x^3+x^2+x+2 = (Ax+B)(x^2+1) + (Dx+F)(x^2+2)$$

$$\Rightarrow x^3+x^2+x+2 = Ax^3+Ax+Bx^2+B+Dx^3+2Dx+Fx^2+2F$$

$$\Rightarrow x^3+x^2+x+2 = (A+D)x^3+(B+F)x^2+(A+2D)x+(B+2F)$$

$\Rightarrow A + D = 1 \cdots (1)$

$\Rightarrow B + F = 1 \cdots (2)$

$\Rightarrow A + 2D = 1 \cdots (3)$

$\Rightarrow B + 2F = 2 \cdots (4)$

From (1) and (3) $\Rightarrow \boxed{D = 0} \Rightarrow \boxed{A = 1}$ From (2)

and (4) $\Rightarrow \boxed{F = 1} \Rightarrow \boxed{B = 0}$

$$\Rightarrow \int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx = \int \frac{x}{x^2+2}dx + \int \frac{dx}{x^2+1} = \frac{1}{2}\ln|x^2+2| + \tan^{-1}(x) + C$$

Example (30): Evaluate the integral $\int \frac{3x^2 + x - 2}{x^3 - x^2 + x - 1} dx$ (H.W)

Case 4: The denominator has one or more quadratic factors, some of which are repeated quadratic factors.

Example (31): Evaluate the integral $\int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2}$$

$$\Rightarrow 1 = A(x^2+1)^2 + (Bx+D)(x^3+x) + (Ex+F)x$$

Solution:

$\Rightarrow 1 = A(x^4 + 2x^2 + 1) + (Bx + D)(x^3 + x) + (Ex + F)x$

$\Rightarrow 1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Dx^3 + Dx + Ex^2 + Fx$

$\Rightarrow 1 = (A + B)x^4 + Dx^3 + (2A + B + E)x^2 + (D + F)x + A$

$\Rightarrow \boxed{A = 1}$ and $\boxed{D = 0}$

$\Rightarrow A + B = 0 \cdots (1) \Rightarrow \boxed{B = -1}$

$\Rightarrow 2A + B + E = 0 \cdots (2) \Rightarrow \boxed{E = -1}$

$\Rightarrow D + F = 0 \cdots (3) \Rightarrow \boxed{F = 0}$

$$\Rightarrow \int \frac{dx}{x(x^2+1)^2} = \int \frac{dx}{x} + \int \frac{-x\,dx}{(x^2+1)} + \int \frac{-x\,dx}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\frac{1}{(x^2+1)} + C$$

Exercises (8.8.2): Evaluate each of the following integrals.

1) $\displaystyle\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx$ 2) $\displaystyle\int \frac{dx}{x^4-9}$ 3) $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$

4) $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}dx$ 5) $\displaystyle\int \frac{3x^2+x-2}{x^3-x^2+x-1}dx$ 6) $\displaystyle\int \frac{x^3}{x^2+x-2}dx$

7) $\displaystyle\int \frac{dx}{(x-1)(x+1)(x^2+1)}$ 8) $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

1.8.9 Special Substitute: Example (32): Evaluate

the integral $\displaystyle\int \frac{dx}{2+2\sqrt{x}}$

Solution:

let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2udu$

$$\Rightarrow \int \frac{dx}{2+2\sqrt{x}} = \int \frac{2udu}{2+2u} = \int \frac{udu}{1+u} = \int du - \int \frac{1}{1+u}du$$

$$= u - \ln|1+u| + C = \sqrt{x} - \ln|1+\sqrt{x}| + C$$

$$\begin{array}{r} 1 \\ 1+u\,\overline{)\,u} \\ \mp u \mp 1 \\ \hline -1 \end{array}$$

Example (33): Evaluate the integral $\displaystyle\int \sqrt{1+e^x}\,dx$

let $u = \sqrt{1+e^x} \Rightarrow u^2 = 1+e^x \Rightarrow e^x = u^2 - 1$

$$\frac{1}{u^2-1}$$

Solution:

$$\Rightarrow x=\ln|u^2-1| \Rightarrow dx=\frac{2u}{u^2-1}du$$

∓@u@2 ±

$$\Rightarrow \int\sqrt{1+e^x}dx=\int u\frac{2u}{u^2-1}du=\int\frac{2u^2du}{u^2-1}=2\int du+2\int\frac{du}{u^2-1}$$

1

1

$$\Rightarrow \int\sqrt{1+e^x}dx=2u+2\int\frac{du}{u^2-1}$$

$$\Rightarrow \frac{1}{u^2-1}=\frac{1}{(u-1)(u+1)}=\frac{A}{(u-1)}+\frac{B}{(u+1)}$$

$$\Rightarrow 1=Au+A+Bu-B$$

$\Rightarrow 1 = (A + B)u + (A - B)$

$\Rightarrow A + B = 0\cdots(1) \Rightarrow A = -B$

$\Rightarrow A - B = 1\cdots(2) \Rightarrow -2B = 1$ $\boxed{B = \frac{-1}{2}} \Rightarrow \boxed{A = \frac{1}{2}} \Rightarrow$

Z du Z 1du $\frac{-\frac{1}{2}du}{}$ = − ln |u − 1|− −   Z1   1

⇒ ____ = ____2   +   ln|u + 1| + C u2 − 1   u − 1 u +

1   2   2

√

∴Z √uu −+ 11 + C = 2√1 + ____ ex + ln√1 +1 + eexx −+ 11 + C

1 + exdx = 2u + ln

Exercises (8.9.1): Evaluate the following integrals.

1) $\int\frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$ 2) $\int\frac{dx}{1+\sqrt{x-2}}$

1.8.10 Substitute by $z=\tan\left(\frac{x}{2}\right)$:

Assume that $z=\tan\left(\frac{x}{2}\right)$

Since $\cos^2(\frac{x}{2})=\frac{1+\cos(x)}{2} \Rightarrow 2\cos^2(\frac{x}{2})=1+\cos(x)$

$$\Rightarrow \cos(x) = 2\cos^2\left(\tfrac{x}{2}\right) - 1 = \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+\tan^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+z^2} - 1$$

$$\Rightarrow \boxed{\cos(x) = \frac{1-z^2}{1+z^2}}$$

Also

$$\sin(x) = 2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right) = \frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \times \cos^2\left(\tfrac{x}{2}\right)$$

Since

$$= 2\tan\left(\tfrac{x}{2}\right) \times \frac{1}{\sec^2\left(\frac{x}{2}\right)} = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow \boxed{\sin(x) = \frac{2z}{1+z^2}}$$

Also

$z = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1}(z) \Rightarrow x = 2\tan^{-1}(z)$

$$\Rightarrow \boxed{dx = \frac{2dz}{1+z^2}}$$

Example (34): Evaluate the integral $\int \frac{dx}{1+\cos(x)}$ Solution:

$$\int \frac{dx}{1+\cos(x)} = \int \frac{1}{1+\frac{1-z^2}{1+z^2}} \times \frac{2dz}{1+z^2} = \int \frac{1}{\frac{1+z^2+1-z^2}{1+z^2}} \times \frac{2dz}{1+z^2} = \int \frac{1}{\frac{2}{1+z^2}} \times \frac{2dz}{1+z^2}$$

$$= \int \frac{1+z^2}{2} \times \frac{2dz}{1+z^2} = \int dz = z + C = \tan\left(\frac{x}{2}\right) + C$$

Example (35): Evaluate the integral $\int \frac{dx}{1-\sin(x)+\cos(x)}$

Solution:

$$\int \frac{dx}{1-\sin(x)+\cos(x)} = \int \frac{\frac{2dz}{1+z^2}}{1-\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2-2z+1-z^2}{1+z^2}} = \int \frac{2dz}{2-2z} = \int \frac{dz}{1-z}$$

$$= -\ln|1-z| + C = -\ln\left|1-\tan\left(\frac{x}{2}\right)\right| + C$$

Exercises (8.10.1): Evaluate the following integrals.

1) $\int \frac{dx}{2+\sin(x)}$ 2) $\int \frac{dx}{\sin(x)+\tan(x)}$

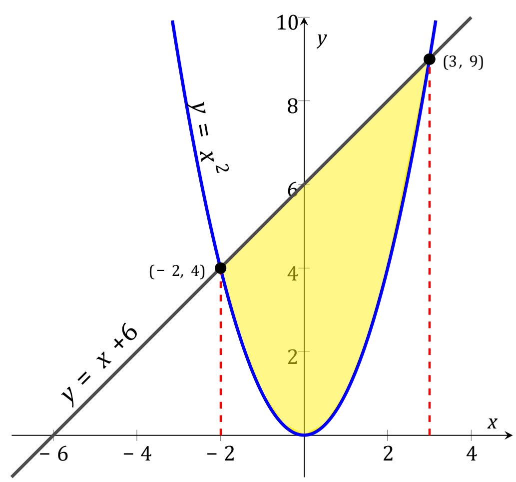
1.9 Application of Definite Integrals

1.9.1 Area Between Curves

Definition (9.1.1): If $f(x)$ and $g(x)$ are continuous functions on the interval $[a,b]$ and $f(x) \geq g(x)$ for all x in $[a,b]$ then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $\underline{a}$ to $\underline{b}$ is the integral of $(f - g)$ from $\underline{a}$ to $\underline{b}$ *i.e*:-

$$A = Area = \int_a^b (f(x) - g(x))\, dx$$

Example (1): Find the area of the region bounded by $y = x + 6$ and the curve $y = x^2$.



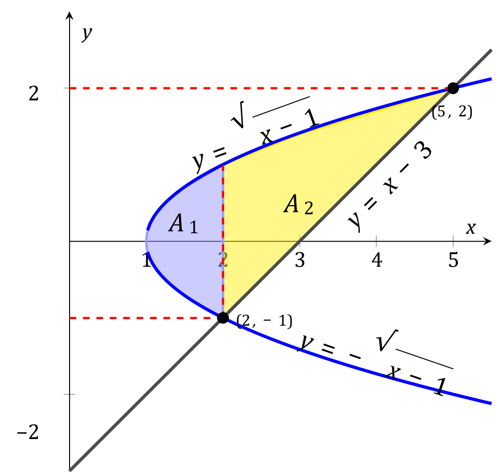
Solution:

$x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3$ and $x = -2$

$$\therefore A = \int_{-2}^{3} \left((x+6) - x^2\right) dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$A = \left(\frac{9}{2} + 18 - \frac{27}{3}\right) - \left(\frac{4}{2} - 12 + \frac{8}{3}\right) = \frac{125}{6}$$ unit area.

Example (2): Find the area of the region bounded by $y^2 = x - 1$ and the line $y = x - 3$.



Solution:

$y^2 + 1 = y + 3 \Rightarrow y^2 - y + 1 - 3 = 0 \Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y - 2)(y + 1) = 0$

$\Rightarrow y = 2 \Rightarrow x = 5$

$\Rightarrow y = -1 \Rightarrow x = 2$

$$\therefore A = \int_{-1}^{2} \left((y+3) - (y^2+1)\right) dy = \int_{-1}^{2} \left(-y^2 + y + 2\right) dy$$

$$= \left[\frac{-y^3}{3} + \frac{y^2}{2} + 2y\right]_{-1}^{2} = \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \left(\frac{-8}{3} + 6\right) - \left(\frac{1}{3} - \frac{3}{2}\right) = \frac{27}{6}$$

<u>*OR :*</u>

unit area.

$$A = \int_{1}^{2} \left(\sqrt{x-1} - (-\sqrt{x-1})\right) dx + \int_{2}^{5} \left(\sqrt{x-1} - (x-3)\right) dx = \frac{27}{6}$$ unit area.

Example (3): Find the area of the region bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{2}$.

Solution:

The point of intersection occur when

$$\sin(x) = \cos(x) \Rightarrow \frac{\sin(x)}{\cos(x)} = 1 \Rightarrow \tan(x) = 1 \Rightarrow x = \frac{\pi}{4}$$

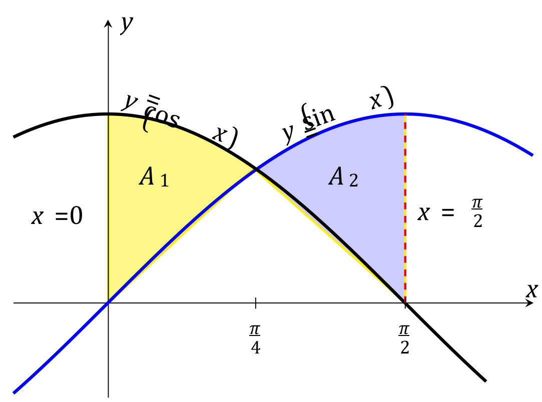
unit area.

$$A_1 = \int_{0}^{\frac{\pi}{4}} (\cos(x) - \sin(x))\, dx = [\sin(x) + \cos(x)]_{0}^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = \frac{2-\sqrt{2}}{\sqrt{2}}$$

$$A_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin(x) - \cos(x))\, dx = [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= (0-1) - \left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -1 + \frac{2}{\sqrt{2}} = \frac{2-\sqrt{2}}{\sqrt{2}}$$



unit area.

$\Rightarrow A = A_1 + A_2 = \dfrac{4 - 2\sqrt{2}}{\sqrt{2}}$ unit area.

Exercises (9.1.1):

1) Find the area between the curve $y = \cos(x)$ and $y = -\sin(x)$ from 0 to $\frac{\pi}{2}$.

2) Find the area of the region bounded above by $y = x^2 + 1$ and below by $y = x - 6$ from $x = -1$ to $x = 3$.

3) Find the area of the region bounded by $y = x^2 - 3x + 12$ and $y = 18 + x - x^2$.

4) Find the area of the region between $y = x + 1$ and $y = 7 - x$ from $x = 2$ to $x = 5$.

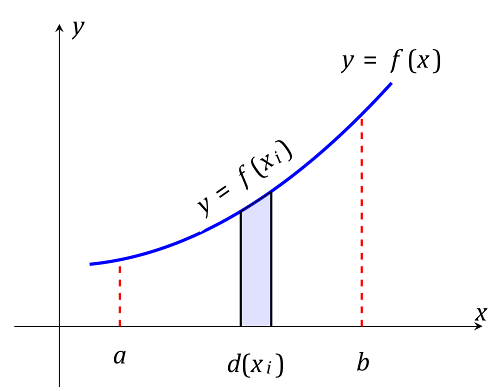
5) Find the area of the region between $y = 3x^3 - x^2 - 10x$ and $y = -x^2 + 2x$.

6) Find the area of the region bounded by $y = x^3$ and the line $y = 2x$.

1.9.2 Area Under the Curve

Definition (9.2.1): If $f(x)$ is positive continuous function on $[a,b]$. Then the area of region bounded by the curve $f(x)$ and $x - axis$ and the lines $x = a$ and $x = b$ is

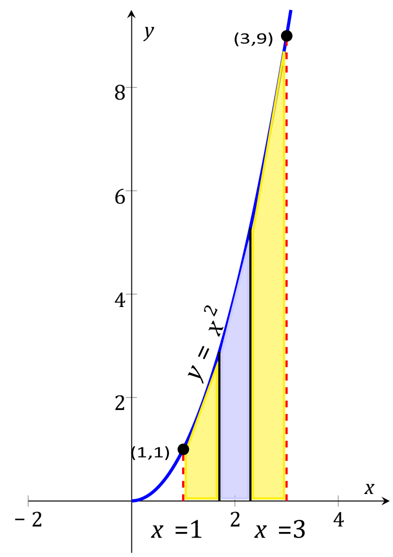
$$A = Area = \int_a^b f(x)dx$$



Remark (9.2.1): If $f(x)$ is negative and continuous on $[a,b]$. Then the area of region bounded by the curve $f(x)$ and $x - axis$ and the lines $x = a$ and $x = b$ is

$$A = Area = -\int_a^b f(x)dx$$

Example (4): Find the area of the region bounded by $y = x^2$ and $x - axis$ and the lines $x = 1$ and $x = 3$.



$$A = \int_a^b f(x)dx = \int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

Solution:

unit area.

Example (5): Find the area of the region by up the $x - axis$ and under the curve $y = 4x - x^2$.

Solution:

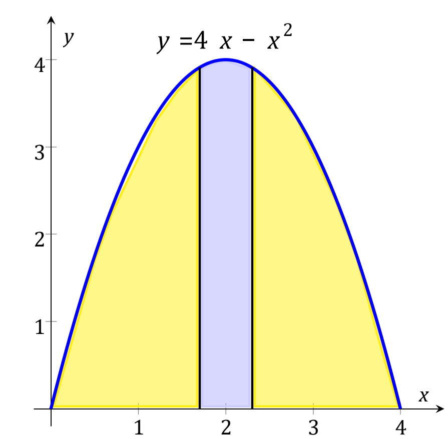
We find the intersection point with $x - axis$.

$y = 0 \Rightarrow 4x - x^2 = 0 \Rightarrow x(4 - x) = 0 \Rightarrow x = 0$ or

$x = 4$

$$\Rightarrow A = \int_0^4 (4x - x^2)dx = \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4 = \frac{32}{3}$$ unit

area.



Exercises (9.2.1):

Find the area bounded by the curve $x = 8 + 2y - y^2$ and $y - axis$ and the lines $y = 3$ and $y = -1$

1.9.3 Area of the Surface

Definition (9.3.1): If the function $f(x)$ has a continuous first derivative throughout the interval $a \le x \le b$, the area of the surface generated by revolving the curve $y = f(x)$ about the $x-axis$ is the number

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Remark (9.3.1): If the function $x = g(y)$ has a continuous first derivative throughout the interval $c \le y \le d$, the area of the surface S generated by revolving the curve $x = g(y)$ about the $y-axis$ is the number

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example (6): Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ about the $x-axis$.

Solution:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = 2\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\Rightarrow S = \int_1^2 2\pi 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}dx = \left[4\pi\left(\frac{2}{3}(x+1)^{\frac{3}{2}}\right)\right]_1^2 = \frac{8\pi}{3}\left(\sqrt{3^3} - \sqrt{2^3}\right)$$

$$= \frac{8\pi}{3}\left(\sqrt{27} - \sqrt{8}\right)$$ unit area.

Example (7): Find the area of the surface generated by revolving the curve $y = 1 - x$,

$0 \le y \le 1$ about the $y - axis$.

Solution:

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$y = 1 - x \Rightarrow x = 1 - y \Rightarrow \frac{dx}{dy} = -1 \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 2$$

$$\Rightarrow S = \int_0^1 2\pi(1-y)\sqrt{2}dy = 2\sqrt{2}\pi\int_0^1 (1-y)dy = 2\sqrt{2}\pi\left[y - \frac{y^2}{2}\right]_0^1 = 2\sqrt{2}\pi\left[(1-\frac{1}{2}) - 0\right]$$

$$= 2\sqrt{2}\pi \times \frac{1}{2} = \sqrt{2}\pi$$ unit area.

Example (8): The circle $x^2 + y^2 = 9$ revolving about $x - axis$ find the area of the surface generated by the revolving.

Solution:

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = \sqrt{9-x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{9-x^2}}$$

$$\Rightarrow S = \int_{-3}^3 2\pi\sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}dx = \int_{-3}^3 2\pi\sqrt{9-x^2}\sqrt{\frac{9-x^2+x^2}{9-x^2}}dx$$

$$= \int_{-3}^3 2\pi\sqrt{9-x^2}\sqrt{\frac{9}{9-x^2}}dx = \int_{-3}^3 2\pi\sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}dx = \int_{-3}^3 6\pi dx = [6\pi x]_{-3}^3$$

$= 18\pi + 18\pi = 36\pi$ unit area.

Exercises (9.3.1):

Find the area of the surface generated by revolving the curve $y = \cos(x),\ 0 \le x \le \frac{\pi}{2}$ about the $x - axis$.

1.9.4 Length of an Arc of a Curve

Definition (9.4.1): If the function $f(x)$ has a continuous first derivative throughout the interval $a \le x \le b$ the length of the curve $y = f(x)$ from $\underline{a}$ to $\underline{b}$ is the number:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Remark (9.4.1):

1) If $x = g(y)$, $c \le y \le d$ then $$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

2) If $x = h(t)$, $y = g(t)$, $t_1 \le t \le t_2$ then $$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example (9): Find the length of the curve; $\frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1$, $0 \le x \le 1$.

Solution:

$$y' = 2\sqrt{2}x^{\frac{1}{2}} \Rightarrow 1 + (y')^2 = 1 + 8x$$

$$\therefore L = \int_0^1 \sqrt{1+8x}dx = \left[\frac{1}{8}\frac{(1+8x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \left[\frac{1}{12}\sqrt{(1+8x)^3}\right]_0^1 = \frac{1}{12}\left(\sqrt{729} - \sqrt{1}\right)$$

$$= \frac{1}{12}(27-1) = \frac{26}{12} = \frac{13}{6}$$ unit length.

Example (10): Find the length of the curves; $y = 1 - \cos(\theta)$, $x = \theta - \sin(\theta)$, $0 \le \theta \le 2\pi$.

Solution:

$$L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\theta_1}^{\theta_2} \sqrt{(1-\cos(\theta))^2 + (\sin(\theta))^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)} d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos(\theta)} d\theta$$

$$= 2\int_0^{2\pi} \sqrt{\sin^2\left(\frac{1}{2}\theta\right)} d\theta = 2\int_0^{2\pi} \sin\left(\frac{1}{2}\theta\right) d\theta = \left[-4\cos\left(\frac{1}{2}\theta\right)\right]_0^{2\pi}$$

$= -4(-1 - 1) = 8$ unit length.

Exercises (9.4.1):

1) Find the length of the curve; $y = e^x$ from $x = 1$ to $x = 2$. 2) Find the length of the curve; $y = x^2$ such that $0 \leq x \leq 1$.

1.9.5 Volumes

Solids of revolution are solids whose shapes can be generated by revolving plan regions about axes.

i. Disk Method

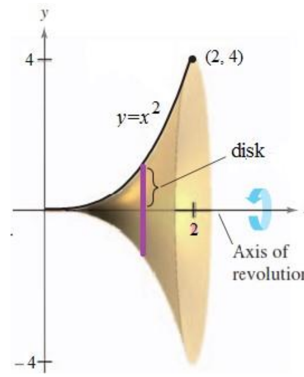
Definition (9.5.1): Volume of a solid of revolution (Rotation about the $x - axis$). the volume of the solid generated by revolving the region between the graph of a continuous function $y = f(x)$ and the $x - axis$ from $x = a$ to $x = b$ about the $x - axis$ is:

$$V = Volume = \int_a^b \pi \left(f(x)\right)^2 dx \qquad \cdots (1)$$

• Volume of a solid of revolution (Rotation about the $y - axis$) is:

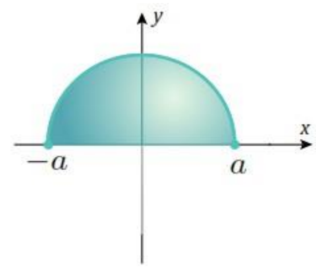
$$V = Volume = \int_a^b \pi (f(y))^2 \, dy \quad \cdots (2)$$

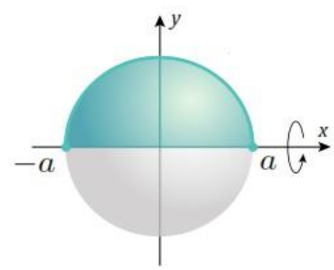
Example (11): The region between the curve $y = x^2$, $x = 0$, $x = 2$ and $x - axis$, is revolved about $x - axis$. Find its volume.



$$V = \pi \int_a^b (f(x))^2 \, dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$$

Example (12): The region enclosed by the semicircle $\overline{a - x}$ and $x - axis$ is revolved about the $x - axis$ to generated a sphere. Find the volume of the sphere.





$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx = \pi \int_{-a}^{a} (a^2 - x^2) dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \pi \left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \pi \left[\frac{2a^3}{3} + \frac{2a^3}{3}\right] = \frac{4}{3}\pi a^3$$

Solution:

$y = \sqrt{\ }$ 2 2

Solution:

ملاحظة:

١- اذا كانت الشريحة عمودية على x (dx) والدوران حول x تستخدم طريقة القرص (المعادلة (١)).

٢- اذا كانت الشريحة عمودية على y (dy) والدوران حول y تستخدم طريقة القرص (المعادلة (٢)).

Exercises (9.5.1):

1) The region between the curve $x = \frac{1}{\sqrt{y}}\ ,\ 1 \le y \le 4$ is revolved about the $y - axis$ to generate a solid. Find the volume of the solid.

2) Find the volume generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.

ii. Washer Method

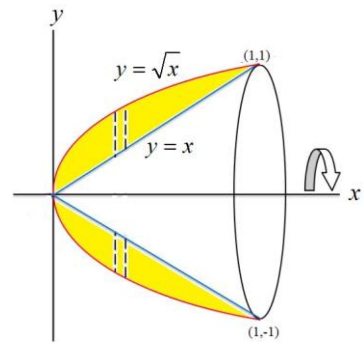
Definition (9.5.2): Let f and g be continuous and nonnegative on $[a,b]$, and suppose that $f(x) \ge g(x)$ for all x in the interval $[a,b]$, then the volume of the solid generated by revolving the region bounded above by $y = f(x)$, below by $y = g(x)$ and on the sides by the lines $x = a$ and $x = b$ about the $x - axis$ is:

$$V = Volume = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx \qquad \cdots (1)$$

• Volume of a solid of revolution (Rotation about the $y - axis$) is:

$$V = Volume = \int_a^b \pi \left[(f(y))^2 - (g(y))^2 \right] dy \qquad \cdots (2)$$

Example (13): The area between the curve $y = x$ and $y = x$ is revolved about $x - axis$ to generated a solid.



Find the volume of the solid.

$$V = \pi \int_0^1 \left(y_2^2 - y_1^2\right) dx = \pi \int_0^1 (x - x^2) dx = \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \pi \left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0)\right] = \frac{\pi}{6}$$

Solution:

Example (14): The region bounded by the parabola $y = x^2$ and the line $y = 2x$ is revolved about the line $x = 2$ parallel to the $y - axis$. Find the volume of the solid.

Solution:

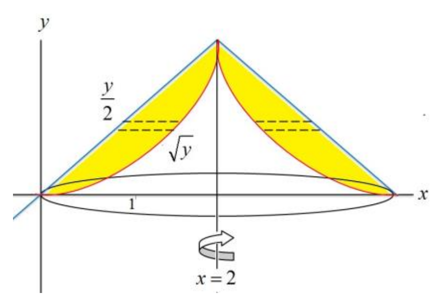
$x = \frac{y}{2}$ and $x = \sqrt{y}$

$\frac{y}{2} = \sqrt{y} \Rightarrow \frac{y^2}{4} = y \Rightarrow 4y = y^2 \Rightarrow 4y - y^2 = 0 \Rightarrow y = 0$ and $y = 4$

$$V = \pi \int_0^4 \left(R^2(y) - r^2(y)\right) dy$$

$$= \pi \int_0^4 \left(\left(2 - \frac{y}{2}\right)^2 - (2 - \sqrt{y})^2\right) dy$$

$$= \pi \int_0^4 \left(\frac{y^2}{4} - 3y + 4\sqrt{y}\right) = \pi \left[\frac{y^3}{12} - \frac{3y^2}{2} + \frac{8}{3} y^{\frac{3}{2}}\right]_0^4 = \frac{8}{3}\pi$$



Exercises (9.5.2):

1) Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

2) Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0,2]$ is revolved about the $x - axis$.

iii. Cylindrical Shell (Shell Method)

Definition (9.5.3): Let $y = f(x)$ be continuous and nonnegative on the interval $[a,b]$ $(0 \leq a < b)$, and let R be the region that is bounded above by $y = f(x)$, below by $x - axis$, and on the sides by the lines $x = a$ and $x = b$. Then, the volume of the solid generated by revolving the region R about the $y - axis$ is given by:

$$V = Volume = \int_a^b 2\pi x f(x)\, dx \qquad \cdots (1)$$

- $x = g(y)$, $c \leq y \leq d$ about $x - axis$ is;

$$V = Volume = \int_c^d 2\pi y g(y)\, dy \qquad \cdots (2)$$

ملاحظة:

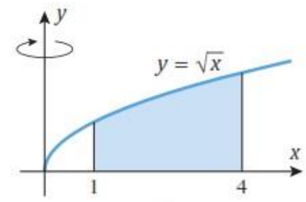
1- اذا كانت الشريحة عمودية على x (dx) والدوران حول y نستخدم المعادلة (1).

2- اذا كانت الشريحة عمودية على y (dy) والدوران حول x نستخدم المعادلة (2).

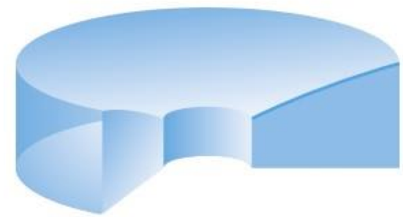
Example (15): Find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the $x - axis$ is revolved about the $y - axis$.

Solution:

First sketch the region (Figure a); then imagine revolving it about the $y-axis$ (Figure b).



(a)


(b)

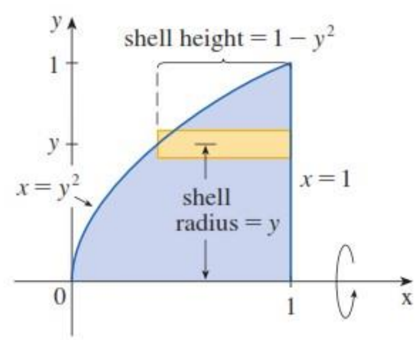
$$V = 2\pi \int_a^b x f(x) dx = 2\pi \int_1^4 x\sqrt{x} dx$$

$$= 2\pi \int_1^4 x^{\frac{3}{2}} dx = 2\pi \left[\frac{2}{5} x^{\frac{5}{2}}\right]_1^4 = \frac{124\pi}{5}$$

Example (16): Use cylindrical shells to find the volume of the solid obtained by rotating about the $x-axis$ the region under the curve $y = \sqrt{x}$ from 0 to 1.

Solution:

If $y = \sqrt{x} \Rightarrow x = y^2$



$$\therefore V = 2\pi \int_0^1 y(1-y^2) dy = 2\pi \int_0^1 (y - y^3) dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

Exercises (9.5.3):

1) The region bounded by the parabola $y = x^2$, the $y-axis$ and the line $y = 1$ is revolved about the line $x = 2$ to generate a solid. Find the volume of the solid.

2) The region bounded by the curve $y = x^3$, the $x-axis$ and the line $x = 1$ is revolved about $x-axis$ to generate a solid. Find the volume of the solid.

2

Sequences and Series

2.1 Sequences:

Definition (2.1.1): An infinite sequence of numbers is a function whose domain is the set of all positive integers.

i.e : A function f : $Z^+ \longrightarrow X$ where X is any set, called a sequence in X.

Remark (2.1.1):

1) Since the sequence is a function and has domain Z^+, then we can to say the sequence by the set: $\{(n, f(n))\ / n \in \mathbb{Z}^+\}$
2) Since the domain all the sequence is the set Z^+, then : $\{(n, f(n))\ / n \in \mathbb{Z}^+\} = \{f(n)\}$
3) If $f(n) = a_n$, then the sequence $\{f(n)\}$ is written as: $\{a_n\} = \{a_1, a_2, \cdots, a_n, \cdots\}$ Example (1):

$$f(n) = \frac{1}{n+1}, n \in \mathbb{Z}^+$$

$$\Rightarrow \left\{\frac{1}{n+1}\right\}_{n=1}^{+\infty} = \left\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n+1}, \cdots\right\} = \{f(n)\}_{n=1}^{+\infty}$$

The number $f(n)$ is the $n - th$ terms of the sequence or the term with index n.

Example (2): $f(n) = \cos\left(\frac{n\pi}{2}\right), n \in \mathbb{Z}^+$

$$\Rightarrow \left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{+\infty} = \{0, -1, 0, 1, 0, -1, \cdots\}$$

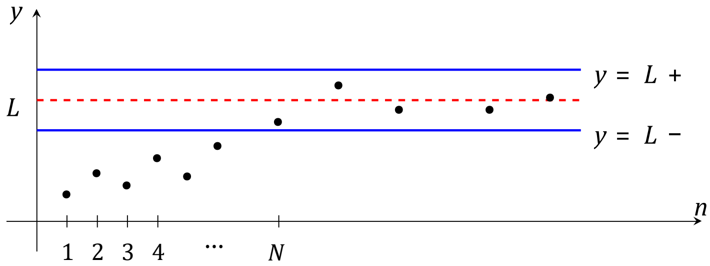
Example (3):

The terms	$n-th$ terms	The sequence
0, 1, 2, 3, ⋯	$n-1$	$\{n-1\}_{n=1}^{+\infty}$
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$	$\frac{1}{n}$	$\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$
$1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \cdots$	$(-1)^{n+1}\frac{1}{n}$	$\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty}$
$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$	$\frac{n-1}{n}$	$\left\{\frac{n-1}{n}\right\}_{n=1}^{+\infty}$
$0, \frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \cdots$	$(-1)^{n+1}\frac{n-1}{n}$	$\left\{(-1)^{n+1}\frac{n-1}{n}\right\}_{n=1}^{+\infty}$
3,3,3,3,⋯	3	$\{3\}_{n=1}^{+\infty}$
$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \cdots$	$\frac{n}{2n+1}$	$\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

Theorem (2.1.1):

The sequence $\{a_n\}$ is convergent if $\lim_{n\to\infty} a_n = L$ (the limit is exist and finite). If no such limit exists, we say that $\{a_n\}$ is divergent.

Example (4): Show that $\left\{\dfrac{4n^2}{2n^2+1}\right\}_{n=1}^{+}$ is convergent.



Solution:

$$\lim_{n\to\infty}\frac{4n^2}{2n^2+1}=\lim_{n\to\infty}\frac{\frac{4n^2}{n^2}}{\frac{2n^2}{2}+\frac{1}{n^2}}=\lim_{n\to\infty}\frac{4}{2+\frac{1}{n^2}}=\frac{4}{2+\frac{1}{\infty}}=\frac{4}{2+0}=2$$

∴ The sequence is convergent.

Example (5): Show that whether $\left\{\dfrac{e^n}{n}\right\}_{n=1}^{+\infty}$ convergent or not.

Solution:

$\lim_{n\to\infty}\dfrac{e^n}{n}=\dfrac{\infty}{\infty}$ we will use (L'Hôpital's Rule) we get $\lim_{n\to\infty}\dfrac{e^n}{1}=\dfrac{\infty}{1}=\infty$

∴ The sequence is divergent.

Theorem (2.1.2): Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequence such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$

and are finite, then:

1) $\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n = ka$; k is constant.

2) $\lim_{n\to\infty}(a_n \mp b_n) = \lim_{n\to\infty} a_n \mp \lim_{n\to\infty} b_n = a \mp b$

3) $\lim_{n\to\infty}(a_n\times b_n)=\lim_{n\to\infty}a_n\times\lim_{n\to\infty}b_n=a\times b$

4) $\lim_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=\dfrac{\lim_{n\to\infty}a_n}{\lim_{n\to\infty}b_n}=\dfrac{a}{b}$; $\lim_{n\to\infty}b_n\neq 0$

5) If $\lim_{n\to\infty}a_n=\infty\Rightarrow\lim_{n\to\infty}\left(\dfrac{1}{a_n}\right)=0$

6) $\lim_{n\to\infty} a_n^r = \left(\lim_{n\to\infty} a_n\right)^r = a^r, \forall r$ is real number such that a^r is exist.

7) $\lim_{n\to\infty} r^{a_n} = r^{\left(\lim_{n\to\infty} a_n\right)} = r^a, \forall r$ is real number.

Example (6): Test the following sequences are convergent or not.

1) $\left\{2^{\frac{1}{n}}\right\}_{n=1}^{+\infty}$ 2) $\left\{\sqrt{\frac{n+1}{n}}\right\}_{n=1}^{+\infty}$ 3) $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{+\infty}$

4) $\left\{\frac{5}{n^2}\right\}_{n=1}^{+\infty}$ 5) $\left\{\frac{4-7n^6}{n^6+3}\right\}_{n=1}^{+\infty}$ 6) $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$

7) $\left\{\frac{2n}{5n+1}\right\}_{n=1}^{+\infty}$ 8) $\{2n\}_{n=1}^{+\infty}$ 9) $\left\{\left(1-\frac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

Solution:

1) $\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\left(\lim_{n\to\infty}\frac{1}{n}\right)} = 2^0 = 1$

∴ The sequence is *convergent*.

2) $\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{n}} = \sqrt{1} = 1$

∴ The sequence is *convergent*.

3) $\lim_{n\to\infty} \frac{\ln(n)}{n} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{0}{1} = 0$

∴ The sequence is *convergent*.

4) $\lim_{n\to\infty} \frac{5}{n^2} = \frac{5}{\infty} = 0$

∴ The sequence is *convergent*.

5) $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{\frac{4}{n^6}-\frac{7n^6}{n^6}}{\frac{n^6}{n^6}+\frac{3}{n^6}} = \frac{0-7}{1+0} = -7$

∴ The sequence is *convergent*.

6)

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)=\lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\lim_{n\to\infty}\sqrt{n+1}+\lim_{n\to\infty}\sqrt{n}}$$

$$=\frac{1}{\infty+\infty}=\frac{1}{\infty}=0$$

∴ The sequence is *convergent*.

7) $$\lim_{n\to\infty}\frac{2n}{5n+1}=\lim_{n\to\infty}\frac{2}{5+\frac{1}{n}}=\frac{2}{5+0}=\frac{2}{5}$$

∴ The sequence is *convergent*.

8) $$\lim_{n\to\infty}2n=\infty$$

∴ The sequence is *divergent*.

9)

$$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{n}=\lim_{n\to\infty}e^{\ln\left(1-\frac{2}{n}\right)^{n}}=e^{\lim_{n\to\infty}\ln\left(1-\frac{2}{n}\right)^{n}}=e^{\lim_{n\to\infty}n\ln\left(1-\frac{2}{n}\right)}$$

$$=e^{\lim_{n\to\infty}\frac{\ln\left(1-\frac{2}{n}\right)}{\frac{1}{n}}}=e^{\frac{0}{0}}$$

$$=e^{\lim_{n\to\infty}\frac{\ln\left(\frac{n-2}{n}\right)}{\frac{1}{n}}}=e^{\lim_{n\to\infty}\frac{\frac{n}{n-2}\times\frac{n-n+2}{n^2}}{\frac{-1}{n^2}}}=e^{\lim_{n\to\infty}\frac{-2n}{n-2}}=e^{\frac{-2}{1}}=e^{-2}$$

∴ The sequence is *convergent*.

Example (7): Show that $\left\{\frac{n^2}{2n+1}\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ is convergent.

Solution:

$$\frac{n^2}{2n+1}\sin\left(\frac{\pi}{n}\right)=\left(\frac{n}{2n+1}\right)\left(n\sin\left(\frac{\pi}{n}\right)\right)$$

let $a_n=\frac{n}{2n+1}$ and $b_n=n\sin\left(\frac{\pi}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \Rightarrow \{a_n\} \text{ is } convergent$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n\sin\left(\frac{\pi}{n}\right)$$

let $m = \frac{\pi}{n} \Rightarrow n = \frac{\pi}{m}$

$\Rightarrow$ If $n \to \infty \Rightarrow m \to 0$

$$\therefore \lim_{n\to\infty} b_n = \lim_{n\to\infty} n\sin\left(\frac{\pi}{n}\right) = \lim_{m\to 0} \frac{\pi}{m}\sin(m) = \pi \lim_{m\to 0} \frac{\sin(m)}{m} = \pi \times 1 = \pi$$

$$\Rightarrow \{b_n\} \text{ is } convergent.$$

$$\therefore \lim_{n\to\infty} \frac{n^2}{2n+1}\sin\left(\frac{\pi}{n}\right) = \frac{1}{2} \times \pi = \frac{\pi}{2} \Rightarrow$$ The sequence is *convergent*.

Theorem (2.1.3):

If a sequence $\{a_n\}$ convergent, then its limit is unique.

Definition (2.1.2): A sequence $\{a_n\}_{n=1}^{\infty}$ is called:

increasing if $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$ (*i.e.*, $a_n \le a_{n+1}, \forall n$). decreasing if

$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$ (*i.e.*, $a_n \ge a_{n+1}, \forall n$).

A sequence that is either increasing or decreasing is said to be monotonic.

Example (8): Explain the following sequences monotonic or not?

1) $\{n\}_{n=1}^{\infty}$ 2) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ 3) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

Solution:

1) $\{n\}_{n=1}^{\infty} = \{1, 2, 3, \cdots\}$

Since $n \leq n+1 \Rightarrow a_n \leq a_{n+1} \Rightarrow \{n\}_{n=1}^{\infty}$ is $increasing$.

Hence the sequence $\{n\}_{n=1}^{\infty}$ is $monotonic$.

2) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$

Since $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$\because n+1 \geq n \Rightarrow \frac{1}{n+1} \leq \frac{1}{n} \Rightarrow a_{n+1} \leq a_n \Rightarrow \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is $decreasing$

Hence the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is $monotonic$.

3) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

$a_n = \frac{(-1)^n}{n}$, $a_{n+1} = \frac{(-1)^{n+1}}{n+1} = -\frac{(-1)^n}{n+1}$

i) If *n* is odd

$\Rightarrow a_n = \frac{-1}{n}$ and $a_{n+1} = \frac{1}{n+1} \Rightarrow a_n \leq a_{n+1}$

$\therefore$ The sequence is *increasing*.

ii) If *n* is even

$\Rightarrow a_n = \frac{1}{n}$ and $a_{n+1} = \frac{-1}{n+1} \Rightarrow a_{n+1} \leq a_n$

$\therefore$ The sequence is *decreasing*.

Hence the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ is *not monotonic*.

Exercises (2.1.1): Show that whether the following sequences are convergent or not?

$\{2\}_{n=1}^{\infty}$ $\left\{n\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ 1)2)3) $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

$\left\{\frac{n^2}{2n+1}\right\}_{n=1}^{\infty}$ $\left\{(-1)^n\frac{2n^3}{n^3+1}\right\}_{n=1}^{\infty}$ 4)5)6) $\left\{\frac{\pi^n}{4^n}\right\}_{n=1}^{\infty}$

$\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=1}^{\infty}$ 7)8) $\left\{\sqrt{n^2+3n}-n\right\}_{n=1}^{\infty}$

Exercises (2.1.2): Write a formula for the $n-th$ term a_n of the following sequence and test the sequence is converge or not?

$$0.7, 0.77, 0.777, 0.7777, \cdots$$

Definition (2.1.3): A sequence $\{a_n\}$ is *bounded above* if there is a number M such that $a_n \le M, \forall n \in \mathbb{Z}^+$ and it is *bounded below* if there is a number m such that $m \le a_n, n \in \mathbb{Z}^+$.

If it is *bounded above* and *below*, then $\{a_n\}$ is *bounded* sequence.

Example (9):

1) $\{n\}_{n=1}^{\infty} = \{1, 2, 3, \cdots\}$ bounded below by 1.

2) $\{1,1,2,2,3,3,\cdots\}$ bounded below by 1.

3) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ bounded below by 0 and bounded above by 1.

4) $\{1,-1,1,-1,\cdots\}$ bounded below by -1 and bounded above by 1.

Theorem (2.1.4):

Every bounded and monotonic sequence is convergent.

Example (10): Show that $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent sequence.

Solution:

Since $n+1 \geq n \Rightarrow \frac{1}{n+1} \leq \frac{1}{n} \Rightarrow a_{n+1} \leq a_n \Rightarrow \{a_n\}$ decreasing $\Rightarrow$ monotonic sequence.

$\because a_n = \frac{1}{n} \leq 1, \forall n \Rightarrow \left\{\frac{1}{n}\right\}$ bounded above by 1. $\because a_n = \frac{1}{n} \geq 0, \forall n \Rightarrow \left\{\frac{1}{n}\right\}$ bounded below by 0 $\Rightarrow \left\{\frac{1}{n}\right\}$ bounded sequence.

$\Rightarrow \left\{\frac{1}{n}\right\}$ convergent sequence (by Theorem (2.1.4)).

Example (11): Show that whether $\left\{(2^n+3^n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ is convergent sequence or not?

Solution:

$$\because 2^n < 3^n \Rightarrow 2^n + 3^n \leq 3^n + 3^n \Rightarrow 2^n + 3^n \leq 2 \times 3^n \Rightarrow (2^n + 3^n)^{\frac{1}{n}} \leq (2 \times 3^n)^{\frac{1}{n}}$$

$$2^n + 3^n \geq 2^n \Rightarrow (2^n+3^n)^{\frac{1}{n}} \geq 2, \forall n \in \mathbb{Z}^+ \Rightarrow \left\{(2^n+3^n)^{\frac{1}{n}}\right\}_n^{\infty}$$ bounded below by 2.

$\Rightarrow (2^n + 3^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} \times 3 = 6$ (since $2^{\frac{1}{n}} \leq 2$)

$\Rightarrow \left\{(2^n+3^n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ bounded above by 6 $\Rightarrow \left\{(2^n+3^n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ bounded sequence.

$$(2^n+3^n)^{\frac{n+1}{n}} = (2^n+3^n)(2^n+3^n)^{\frac{1}{n}} > (2^n+3^n)(3^n)^{\frac{1}{n}} = 3(2^n+3^n) = (3 \times 2^n + 3 \times 3^n)$$

$$> (2 \times 2^n + 3 \times 3^n) = (2^{n+1} + 3^{n+1}) \Rightarrow (2^n+3^n)^{\frac{n+1}{n}} > (2^{n+1} + 3^{n+1})$$

$$\Rightarrow \left((2^n+3^n)^{\frac{n+1}{n}}\right)^{\frac{1}{n+1}} > (2^{n+1}+3^{n+1})^{\frac{1}{n+1}} \Rightarrow (2^n+3^n)^{\frac{1}{n}} > (2^{n+1}+3^{n+1})^{\frac{1}{n+1}} \Rightarrow a_n > a_{n+1}$$

$\Rightarrow$ decreasing sequence $\Rightarrow$ monotonic sequence $\Rightarrow$ convergent sequence.

Theorem (2.1.5):

Let $\{a_n\}$,$\{b_n\}$ and $\{c_n\}$ be three sequences and let $a_n \leq b_n \leq c_n, \forall n$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, where L is constant, then $\lim_{n\to\infty} b_n = L$.

Example (12): Test the convergent of the following

1) $\left\{\frac{\sin(n)}{n}\right\}_{n=1}^{\infty}$ 2) $\left\{\frac{\cos^2(2n)}{4n^2}\right\}_{n=1}^{\infty}$

Solution:

1) Since $-1 \leq \sin(n) \leq 1 \Rightarrow \frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

$$\because \lim_{n\to\infty} \frac{-1}{n} = -\lim_{n\to\infty} \frac{1}{n} = -1 \times 0 = 0$$

$$\because \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0 \Rightarrow \left\{\frac{\sin(n)}{n}\right\}_{n=1}^{\infty}$ convergent sequence to zero. 2)

Since $-1 \leq \cos(2n) \leq 1 \Rightarrow 0 \leq \cos^2(2n) \leq 1 \Rightarrow \frac{0}{4n^2} \leq \frac{\cos^2(2n)}{4n^2} \leq \frac{1}{4n^2}$

$$\Rightarrow 0 \leq \frac{\cos^2(2n)}{4n^2} \leq \frac{1}{4n^2}$$

$\because \lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{4n^2} = 0 \Rightarrow \lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{4n^2} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{\cos^2(2n)}{4n^2} = 0 \Rightarrow \left\{\frac{\cos^2(2n)}{4n^2}\right\}_{n=1}^{\infty}$ convergent sequence to zero.

2.2 Geometric Sequence:

Definition (2.2.1): The sequence of the form $\left\{ar^{n-1}\right\}_{n=1}^{\infty}$ is called geometric sequence,where a, and r are fixed real number and a 6= 0.

$$i.e; \left\{ar^{n-1}\right\}_{n=1}^{\infty} = \left\{a, ar, ar^2, \cdots, ar^{n-1}, \cdots\right\}$$

$b_1 = a$, $b_2 = ar$, $b_3 = ar^2$, ⋯ , $b_n = ar^{n-1}$

Theorem (2.2.1):

If $\left\{ar^{n-1}\right\}_{n=1}^{\infty}$ is geometric sequence then,

converge if $|r| < 1$

$$\{ar^{n-1}\}_{n=1}^{\infty} \text{ is } \begin{cases} converge & \text{if } r = 1 \\ diverge & \text{if } r > 1 \text{ or } r \le -1 \end{cases}$$

Example (1): Test the convergent and write the first three terms of the following sequences.

1) $\left\{5^{n-1}\left(\frac{9}{10}\right)^{n}\right\}_{n=1}^{\infty}$ 2) $\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty}$

Solution:

1)
$$5^{n-1}\left(\frac{9}{10}\right)^{n} = 5^{n-1}\left(\frac{9}{10}\right)^{n-1+1} = 5^{n-1}\left(\frac{9}{10}\right)^{n-1}\left(\frac{9}{10}\right) = \left(\frac{9}{10}\right)\left(\frac{\cancel{5}^{\,1} \times 9}{\cancel{10}^{\,2}}\right)^{n-1}$$

$$= \frac{9}{10}\left(\frac{9}{2}\right)^{n-1} \Rightarrow \left\{5^{n-1}\left(\frac{9}{10}\right)^{n}\right\}_{n=1}^{\infty} = \left\{\frac{9}{10}\left(\frac{9}{2}\right)^{n-1}\right\}_{n=1}^{\infty}$$

$\Rightarrow \left\{\frac{9}{10}\left(\frac{9}{2}\right)^{n-1}\right\}_{n=1}^{\infty}$ diverge (since geometric sequence with $r = \frac{9}{2} = 4.5 > 1$).

$$b_1 = a = \frac{9}{10}$$

$$b_2 = ar = \left(\frac{9}{10}\right)\left(\frac{9}{2}\right) = \frac{81}{20}$$

$$b_3 = ar^2 = \left(\frac{9}{10}\right)\left(\frac{9}{2}\right)^2 = \left(\frac{9}{10}\right)\left(\frac{81}{4}\right) = \frac{729}{40}$$

2) $\frac{1}{2^{n-1}} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} = \left\{\left(\frac{1}{2}\right)^{n-1}\right\}_{n=1}^{\infty}$ converge (since $|r| = \left|\frac{1}{2}\right| < 1$).

$$b_1 = a = 1$$

$$b_2 = ar = 1 \times \frac{1}{2} = \frac{1}{2}$$

$$b_3 = ar^2 = 1 \times \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

2.3 Infinite Series

Definition (2.3.1): Geven a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called an infinite series. The number a_n is called the $n-th$ term of the series.

The sequence $\{S_n\}$ defined as;

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

...

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of the series.

~ If $\{S_n\}$ converge to a limit L then the series converge and that its sum is L.

$$i.e; a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k = L$$

~ If $\{S_n\}$ is not converge then the series diverge.

Example (1): Test the convergent of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots$$

$$S_1 = a_1 = \frac{1}{1\times 2} = \frac{1}{2} = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{1\times 2} + \frac{1}{2\times 3} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

...

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}$$

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} + \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$$\Rightarrow S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\Rightarrow \{S_n\}_{n=1}^{\infty} = \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} = 1 \Rightarrow \{S_n\}_{n=1}^{\infty} = \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$ convergent sequence.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$ converge series to 1.

Example (2): Show that the series $0.333\cdots$ is convergent.

Solution:

$$0.333\cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots$$

$$S_1 = a_1 = \frac{3}{10}$$

$$S_2 = a_1 + a_2 = \frac{3}{10} + \frac{3}{10^2}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} \qquad \cdots(1)$$

multiplying (1) by $\frac{1}{10}$ we get:

$$\frac{1}{10}S_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{n+1}} \qquad \cdots(2)$$

(1)−(2) we get:

$$S_n - \frac{1}{10}S_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\Rightarrow \frac{9}{10}S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right) \Rightarrow S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

$$\Rightarrow \{S_n\}_{n=1}^{\infty} = \left\{\frac{1}{3}\left(1 - \frac{1}{10^n}\right)\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}\left[\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{10^n}\right] = \frac{1}{3}\left[1 - \frac{1}{\infty}\right] = \frac{1}{3}$$

$\Rightarrow \{S_n\}_{n=1}^{\infty} = \left\{\frac{1}{3}\left(1 - \frac{1}{10^n}\right)\right\}_{n=1}^{\infty}$ convergent sequence.

$\Rightarrow$ The series converge to $\frac{1}{3}$.

Theorem (2.3.1):

The necessary condition for the infinite series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ to converge is that $\lim_{n\to\infty} S_n = 0$.

Remark (2.3.1):

1) The converse of theorem a bove is not true.

2) If $\sum_{n=1}^{\infty} S_n$ converge $\Rightarrow \lim_{n\to\infty} S_n = 0$

3) If $\lim_{n\to\infty} S_n = 0 \Rightarrow$ either $\sum_{n=1}^{\infty} S_n$ converge or diverge.

4) If $\lim_{n\to\infty} S_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} S_n$ diverge.

Example (3): Test the converge of the following

1) $\sum_{n=1}^{\infty} \frac{2n+1}{3n+1}$ 2) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ 3) $\sum_{n=1}^{\infty} \frac{1}{n+10}$

Solution:

1) $$\lim_{n\to\infty}\frac{2n+1}{3n+1}=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{3+\frac{1}{n}}=\frac{2}{3}\neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{2n+1}{3n+1}$$ diverge.

2) $$\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1\neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n}{n+1}$$ diverge.

3) $$\lim_{n\to\infty}\frac{1}{n+10}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{10}{n}}=\frac{0}{1+0}=0$$

but $\sum_{n=1}^{\infty}\frac{1}{n+10}$ diverge (we proof later).

Theorem (2.3.2):

Let $\sum a_n$ converge to L_1 and $\sum b_n$ converge to L_2, then 1) $\sum k a_n$ converge to kL_1, where k is constant.

2) $\sum (a_n \mp b_n)$ converge to $L_1 \mp L$[1].

2.4 Geometric Series

Definition (2.4.1): An infinite series of the form:

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[1])

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=0}^{\infty} ar^n$$

is called a geometric series, in which a and r are fixed real number and $a \neq 0$.

Theorem (2.4.1):

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n \begin{cases} \text{converge to } \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverge} & \text{if } |r| \geq 1 \end{cases}$$

Example (1): Test the converge of the following

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ 2) $\displaystyle\sum_{n=0}^{\infty} 5^{n-1}\left(\frac{9}{10}\right)^n$

Solution:

1) $$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \Rightarrow a = 1 \ , \ r = \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \text{ converge to } \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \text{ since } \left(|r| = \left|\frac{1}{2}\right| < 1\right).$$

$$\sum_{n=0}^{\infty} 5^{n-1}\left(\frac{9}{10}\right)^n = \sum_{n=0}^{\infty} 5^n 5^{-1}\left(\frac{9}{10}\right)^n = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{\cancel{5}^{\,1} \times 9}{\cancel{10}^{\,2}}\right)^n = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{9}{2}\right)^n$$

$$\Rightarrow a = \frac{1}{5} \ , \ r = \frac{9}{2}$$

$$\Rightarrow \sum_{n=0}^{\infty} 5^{n-1}\left(\frac{9}{10}\right)^n \text{ diverge since } \left(|r| = \left|\frac{9}{2}\right| = 4.5 > 1\right).$$

Example (2): Explain the geometric series convergent or divergent. Find the partial sums of the series $\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$ Solution:

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{9}\left(1+\frac{1}{3}+\frac{1}{9}+\cdots\right)=\frac{1}{9}\sum_{n=1}^{\infty}\frac{1}{3^{n-1}}=\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow a=\frac{1}{9}, r=\frac{1}{3}$$

Since $r=\frac{1}{3}\Rightarrow$ the geometric series is converge.

$$\Rightarrow \sum_{k=1}^{n} ar^{k-1}=\frac{a}{1-r}=\frac{\frac{1}{9}}{1-\frac{1}{3}}=\frac{1}{6}$$

Example (3): Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}-\frac{4}{2^{n-1}}\right)$ Solution:

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}-\frac{4}{2^{n-1}}\right)=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}-\sum_{n=1}^{\infty}\frac{4}{2^{n-1}}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}-4\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converge to 1.

$\Rightarrow 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converge to (3 × 1) = 3.

Since $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ converge to 2.

$\Rightarrow 4\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ converge to (4 × 2) = 8.

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}-\frac{4}{2^{n-1}}\right)$ converge to (3 − 8) = −5.

Exercises (2.4.1): Test the convergence of the following.

1) $\sum_{n=1}^{\infty}\frac{4}{3^{n-1}}$ 2) $\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}}$

2.5 Test For Convergence

1. *p-Series*:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge if $p > 1$ and diverge if $p \leq 1$.

Example (1): Test the convergence of the following

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ 2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

Solution:

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge since $p = 1$.

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge since $p = 2 > 1$.

3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ diverge since $p = \frac{1}{3} < 1$.

2. Comparison Test:

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two series with non-negative terms, then

~ If $\sum_{n=1} V_n$ is known to be a convergent series then, $\sum_{n=1}^{\infty} U_n$ convergent too if $U_n \leq V_n$, $\forall n$

~ If $\sum_{n=1}^{\infty} V_n$ is known to be a divergent series then, $\sum_{n=1}^{\infty} U_n$ divergent too if $U_n \geq V_n$, $\forall n$ Example (2): Test the convergence of the following

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ 1) 2)3)

!

Solution:

1) $\because n+1>n \ ,\forall n \Rightarrow n(n+1)>n^2 \ ,\forall n \Rightarrow \frac{1}{n(n+1)} \leq \frac{1}{n^2} \ ,\forall n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge by $p-test$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge by comparison test.

2) $\because \ln(n)<n \ ,\forall n \Rightarrow \frac{1}{\ln(n)} \geq \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\ln(n)} \geq \sum_{n=1}^{\infty} \frac{1}{n}$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge by $p-test$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverge by comparison test.

3) $1! = 1 = 2^0$

$2! = 1 \times 2 = 2^1$

$3! = 1 \times 2 \times 3 = 6 > 2^2$

$4! = 1 \times 2 \times 3 \times 4 = 24 > 2^3$

...

$n! > 2^{n-1}$

$\Rightarrow \frac{1}{n!} < \frac{1}{2^{n-1}} \ ,\forall n \in Z^+$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!} < \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converge (geometric series with $r=\frac{1}{2}$). Henceconverge by comparison test.

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad !$$

3. Integration Test:

Suppose that there is a decreasing continuous function $f(x)$, such that $f(x) = U_n$ is the $n-th$ term of the positive series $\sum_{n=1}^{\infty} U_n$, then the series and the integral, $\int_1^{\infty} f(x)dx$ both converge or diverge.

Example (3): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n+10}$ Solution:

$$f(x) = \frac{1}{x+10} \Rightarrow \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x+10}dx = [\ln(x+10)]_1^{\infty} = \ln(\infty+10) - \ln(11) = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+10}$$ diverge by *integration test*.

Exercises (2.5.1): Test the convergence of the following.

1) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2^n}$ 2) $\sum_{n=1}^{\infty} \frac{1}{1+\ln(n)}$ 3) $\sum_{n=1}^{\infty} \frac{2}{2^n+3}$ 4) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

5) $\sum_{n=1}^{\infty} ne^{-n^2}$

6) $\sum_{n=1}^{\infty} \frac{\ln(n)}{2n^3-1}$ 7) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ 8) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ 9) $\sum_{n=1}^{\infty} \frac{n}{n+2}$ 10) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

4. Infinite Series With Alternating Signs:

Theorem (2.5.1):

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - \cdots + a_n - \cdots$ converge if:

1) $|a_{n+1}| < |a_n|$, $\forall n$

2) $\lim_{n\to\infty} |a_n| = 0$

Example (4): Test the convergence of the following series

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ 2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$

Solution:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

1) $|a_1| = |1| = 1$, $|a_1| = |-\frac{1}{2}| = \frac{1}{2}$, $\cdots$

i. $|a_{n+1}| < |a_n|$

$$\frac{1}{2} < 1 \ \& \ \frac{1}{3} < \frac{1}{2} \ \& \ \cdots$$

ii. $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent.

2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$

i. $(2n+1)! < (2n+3)! \Rightarrow \frac{1}{(2n+1)!} > \frac{1}{(2n+3)!} \Rightarrow |a_{n+1}| < |a_n|$, $\forall n$

ii. $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ is convergent.

5. Absolute and Conditional Convergence:

Theorem (2.5.2):

A series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is said to be absolute convergent if the corresponding series of absolute values $\sum_{n=1}^{\infty} |a_n|$, is convergent. But if the series $\sum_{n=1}^{\infty} |a_n|$ diverge while the series $\sum_{n=1}^{\infty} a_n$ converge, the $\sum_{n=1}^{\infty} a_n$ is converge conditionally.

Example (5): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Solution:

Since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent series, but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is diverge by p−*test*.

∴ the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge conditionally.

Example (6): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ Solution:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots + \frac{(-1)^{n+1}}{n^2} - \cdots$$

1) $|a_1| = |1| = 1 \ \& \ |a_2| = \left| \frac{-1}{4} \right| = \frac{1}{4} \ \& \ |a_3| = \frac{1}{9} \ \& \ \cdots \Rightarrow |a_{n+1}| < |a_n|$

2) $\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{1}{n^2} = 0$

∴ The series is convergent. Now,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent by p − *test*.

∴ The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent absolutely.

Remark (2.5.1):

Every absolutely convergent series is convergent (the converse is not true).

6. Ratio Test:

The alternative series $\sum_{n=1}^{\infty} |a_n|$ converge absolutely (and hence convergent) if: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho < 1$. And diverge if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho > 1$. And if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the series may converge or it may diverge (the test provide no information)

Example (7): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^n}\,2}{(n+1)\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{2^n}}\right| = 2\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

∴ The series $\sum_{n=1}^{\infty} \frac{2}{n!}^{n}$ is convergent.

Example (8): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2^{n-1}}{n+4}$ Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^n}{n+5} \times \frac{n+4}{2^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^n}}{n+5} \times \frac{n+4}{\cancel{2^n}\,2^{-1}}\right| = \lim_{n\to\infty}\left|\frac{2(n+4)}{(n+5)}\right|$$

$$= 2\lim_{n\to\infty}\frac{n+4}{n+5} = 2\times 1 = 2 > 1$$

∴ The series $\sum_{n=1}^{\infty} \frac{2}{n+4}^{n-1}$ is divergent.

Example (9): Test the convergence of the series $\sum_{n=0}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \times \frac{3!\,n!\,3^n}{(n+3)!}\right| = \lim_{n\to\infty}\left|\frac{(n+4)\cancel{(n+3)!}}{\cancel{3!}(n+1)\cancel{n!}\,3\,\cancel{3^n}} \times \frac{\cancel{3!}\,\cancel{n!}\,\cancel{3^n}}{\cancel{(n+3)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+4)}{3(n+1)}\right| = \lim_{n\to\infty}\frac{(n+4)}{3(n+1)} = \frac{1}{3}\lim_{n\to\infty}\frac{(n+4)}{(n+1)} = \frac{1}{3}\times 1 = \frac{1}{3} < 1$$

∴ The series $\sum_{n=0}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ is convergent.

Example (10): Find all value of *x* for which the given series converge: $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n\, n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}(n+1)^2} \times \frac{2^n n^2}{(-1)^n (x+1)^n}\right| < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{(-1)(x+1)n^2}{2(n+1)^2}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-1)(x+1)}{2}\left(\frac{n}{n+1}\right)^2\right| < 1$$

$$\Rightarrow \frac{|x+1|}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 < 1 \Rightarrow \frac{|x+1|}{2} \times 1 < 1 \Rightarrow |x+1| < 2$$

$$\Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$$

∞ n n ∞

Solution:

at $x = -3 \Rightarrow \sum_{n=1} \frac{(-1)\ (-2)}{2^n n^2} = \sum_{n=1} \frac{1}{n^2}$ converge by *p* − *test*.

∴ *x* = −3

at $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converge.

∴ *x* = 1

∴ The value is, −3 ≤ *x* ≤ 1

Exercises (2.5.2): Test the convergence of the following.

1) $\sum_{n=1}^{\infty} \frac{(2\,n)!}{n^{100}}$!

2) $\sum_{n=1}^{\infty} \frac{2^n\, n!\, n}{(2\,n)!}$

Exercises (2.5.3): Find all value of x for which the given series converge:

1) $\sum_{n=1}^{\infty} \frac{n\,x^n}{2^n}$ 2) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

2.6 Power Series:

Definition (2.6.1): Power series are defined by:

$$\sum_{n=0}^{\infty} C_n(x-a)^n \ \underline{\underline{\mathcal{OR}}}\ \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

in which the center a and the coefficients $C_0, C_1, C_2, \cdots, C_n, \cdots$ are constant.

Theorem (2.6.1):

Let $\sum_{n=k}^{\infty} C_n(x-a)^n$, be any power series, where $k \geq 0$, then:

1) The series converge only when $x = a$.

2) The series converge for all x.

3) There is a number $< > 0$ such that the series is convergent if $|x-a| < <$ and it is divergent if $|x-a| > <$. And may converge or diverge when $|x-a| = <$. This number $<$ is called the radius of convergence. And the interval of convergence is $(-< + a, < + a)$.

Example (1): Test the convergence of the series $\sum_{n=1}^{\infty} n!(x-1)^n$ Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| = \lim_{n\to\infty}|(n+1)(x-1)| = |x-1|\lim_{n\to\infty}(n+1)$$

$$= |x-1| \times \infty = \infty$$

⇒ The series is divergent for all x 6= 1 and when $x = 1$ the sum of the series is 0. In this case we say the radius of convergence is 0 (< = 0) and the interval of convergence is the point $x = 1$.

Example (2): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\times 0 = 0 < 1$$

The series is convergent for all x and the radius of convergence is (< = ∞) and the interval of convergence is (−∞,∞).

Example (3): Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(3x)^n}{n^3}$

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(n+1)^3}\times\frac{n^3}{(3x)^n}\right| = \lim_{n\to\infty}\left|\frac{(3x)n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\left|\frac{(3x)n^3}{n^3+3n^2+3n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(3x)\frac{n^3}{n^3}}{\frac{n^3}{n^3}+\frac{3n^2}{n^3}+\frac{3n}{n^3}+\frac{1}{n^3}}\right| = \left|\frac{(3x)\times 1}{1+0+0+0}\right| = |3x|$$

The series is convergent if $|3x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow |x - 0^a| < \frac{1}{3}$

and it is divergent if $|3x| > 1 \Rightarrow |x| > \frac{1}{3} \Rightarrow |x - 0^a| > \frac{1}{3}$. The radius of convergence is $\Re = \frac{1}{3}$. To find the interval of convergence we need to examine the end points $x = \frac{1}{3}$ and $x = -\frac{1}{3}$

when $x = \frac{1}{3} \Rightarrow \sum_{n=1}^{\infty}\frac{\left(3\times\frac{1}{3}\right)^n}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3}$ convergent by $p-test$

when $x = -\frac{1}{3} \Rightarrow \sum_{n=1}^{\infty}\frac{\left(3\times\frac{-1}{3}\right)^n}{n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ absolutely convergent and therefore con-vergent.

Hence, the interval of convergence is $[-\frac{1}{3}, \frac{1}{3}]$

Exercises (2.6.1): Test the convergent of the following.

1) $\sum_{n=0}^{\infty} \frac{(-1)^n\, n!\, x^n}{10^n}$ 2) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ 3) $\sum_{n=}^{\infty} \frac{(2\,x-1)^n}{n\,2^n}$

2.7 Representation of Function by Power Series:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

Example (1): Represent the following function by power series:

1) $f(x) = \frac{1}{1+x}$ 2) $f(x) = \frac{2}{2-3x}$

Solution:

1) $f(x) = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + \cdots$ for $|-x| < 1$

$$= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$f(x) = \frac{2}{2-3x} = \frac{1}{1-\frac{3}{2}x} = 1 + \left(\frac{3}{2}x\right) + \left(\frac{3}{2}x\right)^2 + \left(\frac{3}{2}x\right)^3 + \cdots$$

2) $= 1 + \frac{3}{2}x + \frac{9}{4}x^2 + \frac{27}{8}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n x^n$ for $|x| < \frac{2}{3}$

Theorem (2.7.1):

Suppose that the function $f(x)$ can be representation by power series $\sum_{n=0}^{\infty} C_n x^n$, then

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} C_n\, x^n\right) = \sum_{n=0}^{\infty} n\, C_n\, x^{n-1}$$

$$\int f(x)dx = \int \sum_{n=0}^{\infty} C_n\, x^n dx = \sum_{n=0}^{\infty} \frac{C_n\, x^{n+1}}{n+1}$$

1)

2)

Example (2): Represent the following function by power series $f(x) = \tan^{-1}(x)$ Solution:

Since $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$ for $|-x^2| < 1$

$$\Rightarrow \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\Rightarrow \int \frac{1}{1+x^2}dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx$$

$$\Rightarrow \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{2n+1}$$ for $|x| < 1$

2.8 Taylor and Maclaurin Series:

Definition (2.8.1): *Taylor series* of a function $f(x)$ at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$ It is a power series centered at a.

Definition (2.8.2): *Maclaurin series* of a function $f(x)$ is a Taylor series at $x = 0$.

$$i.e : \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

where f^n is the derivative of f with n degree and $f^{(0)} = f$ & $0! = 1$

Example (1): Find the maclaurin expansion of $f(x) = \sin(x)$

Solution:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}$$

$$f^{(0)}(x)=f(x)=\sin(x)\Rightarrow f(0)=\sin(0)=0$$

$$f'(x)=\cos(x)\Rightarrow f'(0)=\cos(0)=1$$

$$f''(x)=-\sin(x)\Rightarrow f''(0)=-\sin(0)=0$$

$$f'''(x)=-\cos(x)\Rightarrow f'''(0)=-\cos(0)=-1$$

$$\vdots$$

$$f^{(4)}(0)=0\ ,\ f^{(5)}(0)=1\ ,\ f^{(6)}(0)=-1\ ,\ \cdots$$

$$\therefore f(x)=\sin(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}$$

$$=\frac{f(0)}{0!}x^{0}+\frac{f'(0)}{1!}x^{1}+\frac{f''(0)}{2!}x^{2}+\frac{f'''(0)}{3!}x^{3}+\cdots$$

$$=0+x+0-\frac{x^{3}}{3!}+0+\frac{x^{5}}{5!}+0-\frac{x^{7}}{7!}+\cdots$$

$$=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots$$

$$\Rightarrow \sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n}\,x^{2n+1}}{(2n+1)!}$$

Example (2): Find the maclaurin expansion of $f(x) = e^{x}$ Solution:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}$$

$$f^{(0)}(x)=f(x)=e^{x}\Rightarrow f(0)=e^{0}=1$$

$$f'(x)=e^{x}\Rightarrow f'(0)=e^{0}=1$$

$$f''(x)=e^{x}\Rightarrow f''(0)=e^{0}=1$$

$$f'''(x)=e^{x}\Rightarrow f'''(0)=e^{0}=1$$

$$\vdots$$

$$f^{(4)}(0) = 1\ ,\ f^{(5)}(0) = 1\ ,\ f^{(6)}(0) = 1\ ,\ \cdots$$

$$\therefore f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \frac{f(0)}{0!} x^0 + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Exercises (2.8.1): Find the maclaurin expansion of the following functions.

1) $f(x) = \cos(x)$ 2) $f(x) = \sin(x^2)$ 3) $f(x) = \frac{\sin(x)}{x}$ 4) $f(x) = x^2 e^x$

Example (3): Find the taylor expansion of $f(x) = \frac{1}{x}$ at $a = 2$

Solution:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n$$

$$f^{(0)}(x) = f(x) = x^{-1} \Rightarrow f(2) = 2^{-1}$$

$$f'(x) = (-1)x^{-2} \Rightarrow f'(2) = (-1)(2)^{-2}$$

$$f''(x) = (-1)(-2)x^{-3} \Rightarrow f''(2) = (-1)(-2)(2)^{-3}$$

$$f'''(x) = (-1)(-2)(-3)x^{-4} \Rightarrow f'''(2) = (-1)(-2)(-3)(2)^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-n)x^{-(n+1)} = (-1)^n\, n!\, x^{-(n+1)} \Rightarrow f^{(n)}(2) = (-1)^n\, n!\, (2)^{-(n+1)}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!\, 2^{-(n+1)}}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$$

Example (4): Find the taylor expansion of $f(x) = e^{-2x}$ at $a = \frac{1}{2}$ Solution:

$$f(x) = e^{-2x} \Rightarrow f\left(\tfrac{1}{2}\right) = e^{-2\times\frac{1}{2}} = e^{-1} = \frac{1}{e}$$

$$f'(x) = (-2)e^{-2x} \Rightarrow f'\left(\tfrac{1}{2}\right) = (-2)e^{-1} = \frac{-2}{e} = \frac{(-1)^1\,2^1}{e}$$

$$f''(x) = (-2)(-2)e^{-2x} \Rightarrow f''\left(\tfrac{1}{2}\right) = (-2)(-2)e^{-1} = \frac{(-1)^2\,2^2}{e}$$

$$f'''(x) = (-2)(-2)(-2)e^{-2x} \Rightarrow f'''\left(\tfrac{1}{2}\right) = \frac{(-1)^3\,2^3}{e}$$

$$\vdots$$

$$f^{(n)}(x) = \underbrace{(-2)(-2)\cdots(-2)}_{n-times}\,e^{-2x} = (-1)^n\,2^n\,e^{-2x}$$

$$\Rightarrow f^{(n)}\left(\tfrac{1}{2}\right) = (-1)^n\,2^n\,e^{-1} = \frac{(-1)^n\,2^n}{e}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{1}{2}\right)}{n!}\left(x-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n\,2^n}{e}}{n!}\left(x-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{e\,(n!)}\,(2\,x-1)^n$$

Exercises (2.8.2):

1) Find the taylor expansion of $f(x) = \dfrac{1}{x}$ at $a = 1$

2) Find the taylor expansion of $f(x) = e^{-x}$ at $a = 0$

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