مفردات المنهج:

1- *التحدب*
2- *المستويات الزائديه*
3- *مبرهنات هان-بناخ*
4- *الفضاءات المعيارية*
5- *مبادئ مترية في الفضاءات المعيارية*
6- *التقارب في الفضاءات المعيارية*
7- *فضاءات بناخ*
8- *الاستمرارية القدية*
9- *فضاءات التحويلات الخطية المقيدة*
10- *فضاءات معياريه قابله للفصل*
11- *مبرهنات الفصل في الفضاءات المعيارية*

Definition:

Let R be a set of real number and " $+$ " addition on R, " $.$ " multipilication on R, then

1- *For any* $a, b \in R$, *then* $a + b \in R$ *and* $a.b \in R$
2- *For any* $a, b, c \in R$ *then*
$(a + b) + c = a + (b + c)$ *and* $(a.b).c = a.(b.c)$
3- *There exists* $0,1 \in R$ *such that*
$a + 0 = 0 + a = a$ *and* $a.1.a = a$
4- *For any* $a \in R$ *and* $a \neq 0$ *there exists* $\frac{1}{a} = a^{-1} \in R$ *and*
$a.\frac{1}{a} = \frac{1}{a}.a = 1$
5- *For any* $a \in R$, *there exists* $-a \in R$ *such that* $a + (-a) = (-a) + a = 0$
6- *For any* $a, b, c \in R$ *then*
$a.(b + c) = a.b + a.c$ *and*
$(b + c).a = b.a + c.a$
7- *For any* $a, b \in R$ *then* $a + b = b + a$ *and* $a.b = b.a$
8- $(\alpha + \beta)a = \alpha a + \beta a$ *for any* $a \in X, \alpha, \beta \in F$
9- $(\alpha.\beta)a = \alpha.(\beta.a)$ *for any* $a \in X, \alpha, \beta \in F$

Hence $(R, +, .)$ *is a field and is said to be a field of real number*

Definition:

Let X be a non-empty set and F be a field then X is said to be vector space over F iff there exists two operations

Addition

$+: X \times X \longrightarrow X$ *and*

Scalar multiplication

$.: F \times X \longrightarrow X$ *between X and F this two operational satisfies conditions*

1- For any $a.b \in X$ then $a + b \in X$
2- For any $a.b \in X$ then $a + b = b + a$
3- For any $a.b.c \in X$ then
   $a + (b + c) = (a + b) + c$
4- there exists $0 \in X$ such that
   $a + 0 = 0 + a = a$ for any $a \in X$ 0 is called zero vector
5- For any $a \in X$ there exists $-a \in X$ such that $a + (-a) = (-a) + a = 0$
6- $\lambda a \in X$ for any $\lambda \in F, a \in X$
7- $\lambda.(a + y) = \lambda.a + \lambda.b$, For any $a, b \in X, \lambda \in F$

Example:

1- Let R be a set of real number and let
   $V = R^n = \{ (a_1, a_2, \dots, a_n)\}$ and $+$, . define on R^n as follows:
   let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n) \in R^n$ then

$(a_1, a_2, \dots, a_n)$ +$(b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ *and let* $\lambda \in R$, $(a_1, a_2, \dots, a_n) \in R^n$ *then,* $\lambda(a_1, a_2, \dots, a_n) = (\lambda a_1, \lambda a_2, \dots, \lambda a_n)$ *show that* V *be a vector space over a field* R.

2- *Let* R *be a set of real number and let* $V = M_{m\times n} = \{A, A \text{ matrix over } R\}$ *and* $+$, .*define on* $A_{m\times n}$ *as follows*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \text{ such that}$$

$$A + B \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

and let $\alpha \in R$ *then* $\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n} \\ \vdots & & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$

show that V *vector space over* R

Theorem:

If X *is a vector space over a field* F *then*

1- $0.x = 0$ *for any* $x \in X$
2- $\lambda.0 = 0$ *for any* $\lambda \in F$
3- $-(\lambda.x) = (-\lambda)x = \lambda(-x)$

4- *For any* $x, y \in X$*, then there exists one element* $z \in X$ *such that* $x + z = y$
5- $\lambda(x - y) = \lambda x - \lambda y$ *for any* $x, y \in X, \lambda \in F$
6- *If* $\lambda x = 0 \longrightarrow \lambda = 0$ *or* $x = 0$
7- *If* $x \neq 0$ *and* $\lambda_1 . x = \lambda_2 . y$ *then* $\lambda_1 = \lambda_2$
8- ***If*** $x \neq 0$ *,* $\lambda \neq 0$ *,* $y \neq 0$ *and* $\lambda x = \lambda y$ *then* $x = y$

Definition:

Let V *be a vector space over a field* F *and* $G \subseteq F$ *,* $A, B \subseteq V$ *we define*$A + B, GA$ *as follows:*

$A + B = \{a + b, a \in A, b \in B\}$

$GA = \{\lambda a, \lambda \in G, a \in A\}$

Example:

Let R *be a set of a real numbers*

And let $A = \{1,2\}$*,* $B = \{-1,7\}$ *,* $C = \{0,4,6\}$ *find* $A + B, A + C$*,* $4A$

$A + B = \{1 + (-1), 1 + 7, 2 + (-1), 2 + 7\} = \{0,8, 1, 9\}$

$A + C = \{ 1,5,7,2,8,6\}$

$4A = \{4.1, 4.2\} = \{4,8\}$

Not:

1- *If* $A = \{a\}$ *(*A *contain only one element we write* $a + B$ *instead of* $\{a\} + B$ *such that set* $a + B$ *to displan* B *by* a
2- *If* $0 \in A$ *then* $B \subseteq A + B$

3- $A + B = \cup_{a \in A}(A + B)$

4- If $G = \{\lambda\}$ we write λA instead of $\{\lambda\}A$ or GA such that $\lambda A = \{x = \lambda a, a \in A\}$ inpartically $-A = (-1)A = \{-a,\ a \in A\}$ we say that A symmetric set if $-A = A$ then $A \cap (-A)$ is symmetric for any $A \subseteq X$

Balanced set

Definition:

A subset A to a space X over a field F is said to be balanced set if $\lambda A \subseteq A$, $\forall \lambda \in F$ and $|\lambda| \leq 1$.

Theorem:

Let A, B are balanced set of a space X over a field F show that whether

1- $A \cap B$, 2- $A \cup B$, 3- $A + B$, 4- αA are balanced sets or not

Proof:

1- Let $\lambda \in F$ and $|\lambda| \leq 1$

Since A, B are balanced set

$\rightarrow \lambda A \subseteq A$ and $\lambda B \subseteq B$

We prove that $\lambda(A \cap B) \subseteq A \cap B$

Now, let $x \in \lambda(A \cap B)$ $\rightarrow$

$x = \lambda y$, where $y \in A \cap B$

Since $y \in A \cap B \rightarrow y \in A \wedge y \in B$

$\rightarrow \lambda y \in \lambda A \wedge \lambda y \in \lambda B$ *(since* $x = \lambda y$*)*

$\rightarrow x \in \lambda A \wedge x \in \lambda B$ *(since* $\lambda A \subseteq A$ *and* $\lambda B \subseteq B$*)*

$\rightarrow x \in A \wedge x \in B$

$\rightarrow x \in A \cap B$

Then we get $\lambda(A \cap B) \subseteq A \cap B \; \forall \lambda \in F$ *and* $|\lambda| \leq 1$

$\rightarrow A \cap B$ *is a balanced set*

2-, 3- and 4- (H.W)

Theorem:

Let A *be a balanced set of a space over* F *and* $\lambda \in F$ *,* $|\lambda| = 1$ *then* $\lambda A = A$ *and every balanced set is a symmetric*

Proof:

suppose that A *is a balanced set*

$\rightarrow \lambda A \subseteq A$ *,* $\lambda \in F$ *and* $|\lambda| \leq 1$

$\rightarrow \lambda A \subseteq A$ *where* $|\lambda| = 1$ *…(1)*

Now, we prove that $A \subseteq \lambda A$ *where* $|\lambda| = 1$

Let $x \in A$

Since, $|\lambda| \neq 0 \rightarrow \lambda \neq 0$

Take $\alpha = \frac{1}{\lambda} \rightarrow |\alpha| = 1$ *becouse* $|\alpha| = \left|\frac{1}{\lambda}\right| = \left|\frac{1}{1}\right| = |1| = 1$

Since A *is a balanced* $\rightarrow \alpha A \subseteq A$

$\rightarrow \alpha x \in A$ $\quad$ ($x \in A, \rightarrow \alpha x \in \alpha A \subseteq A, \rightarrow \alpha x \in A$)

Then , $\lambda(\alpha x) \in \lambda A$

$\rightarrow \lambda.\frac{1}{\lambda}x \in \lambda A$, $\quad \rightarrow x \in \lambda A$

We get $A \subseteq \lambda A$ *…(2)*

From (1) and (2) we get $\lambda A = A$

Now, we prove that A *is a symmetric*

Suppose that A *balanced set and let* $\lambda = -1 \rightarrow |\lambda| = 1$

since, $\lambda A = A \rightarrow -A = A$

$\rightarrow A$ *symmetric*

subspace

Definition :

A non empty sub set M *of a vector space* V *over a field* F *is said to be sub space if* M *is a vector space over* F*.*

Example:'

Let $V = R^3$ *is a vector space over* R *and*

1- $M_1 = \{(x, y, 0), x, y \in R\}$

2- $M_2 = \{(x, 0, y), x, y \in R\}$
3- $M_1 = \{(x, y, z), x, y, z \in R, x \geq 0\}$

Show that whether M_1, M_2 and M_3 are subspace over R or not.

Theorem:

A non empty sub set M of a vector space V over a field F is subspace of V iff

1- $\forall\, x, y \in M, \rightarrow x + y \in M$
2- $\forall\, x \in M, \alpha \in F \rightarrow \alpha x \in M.$

Remark:

For any vector space X over a field F then exists two subspace are $\{0\}$ zero subspace and X are said trivial subspace

**If M is a proper subset of X then M is said to be proper sub space.*

Theorem:

Let M_1 and M_2 are subspace of a vector space V over a field F then

1- $M_1 \cap M_2$ *is a sub space*
2- $M_1 \cup M_2$ *is a sub space iff* $M_1 \subseteq M_2$ *or* $M_2 \subseteq M_1$
3- $M_1 + M_2$ *is a sub space and* $M_1 \subseteq M_1 + M_2 \wedge M_2 \subseteq M_1 + M_2$

Proof

Let $x, y \in M_1 \cap M_2,\ \alpha, \beta \in F$

Since, $x, y \in M_1 \cap M_2 \to x, y \in M_1\ \wedge\ x, y \in M_2$

Now, $x, y \in M_1,\ M_1$ *subspace ,* $\alpha, \beta \in F$

$\to \alpha x + \beta y \in M_1$

Also $x, y \in M_2,\ M_2$ *subspace ,* $\alpha, \beta \in F$

$\to \alpha x + \beta y \in M_2$

Now, $\alpha x + \beta y \in M_1 \wedge \alpha x + \beta y \in M_2$

$$\to \alpha x + \beta y \in M_1 \cap M_2, \alpha, \beta \in F$$

Then $M_1 \cap M_2$ *is a subspace*

Linear comboination

Definition:

Let $v_1, v_2, \dots, v_n \in V$*, where* V *vector space over a field* F *,* $v \in V$ *then* v *is said to be linear combination of* $v_1, v_2, \dots, v_n$ *if there exist* $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ *such that*

$$V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

Example:

Let $V = R^3$ *be a vector space over a field* R *and let* $v_1 = (1,0,0)$*,* $v_2 = (0,1,0)$*,* $v_3 = (0,0,1)$ *and* $v = (2,3,6)$ *show that* v *is a linear combination of* v_1, v_2, v_3

Solution:

$$v = (2,3,6) = 2(1,0,0) + 3(0,1,0) + 6(0,0,1)$$
$$= (2,0,0) + (0,3,0) + (0,0,6) = (2,3,6) = v$$

$\rightarrow v$ *is linear combination of* v_1, v_2, v_3 *where* $\alpha_1 = 2, \alpha_2 = 3, v_3 = 6 \in R.$

Example:

Let $A = \{ (1,2,3), (0,1,2), (0,0,1)\}$ *prove* $[A] = R^3$

Solution

Let $x_1 = (1,2,3),\ x_2 = (0,1,2),\ x_3 = (0,0,1)$ *to prove that every element* $(x, y, z) \in R^3$*is a linear combination of* x_1, x_2, x_3

$$(x, y, z) = \lambda_1\ (1,2,3) + \lambda_2(0,1,2)\lambda_3(0,0,1)$$

$$= (\lambda_1\ , 2\lambda_1\ , 3\lambda_1\ ) + (0, \lambda_2, 2\lambda_2) + (0,0, \lambda_3)$$

$= (\lambda_1\ , 2\lambda_1\ + \lambda_2, 3\lambda_1 +\ 2\lambda_2 + \lambda_3)$

Now, $x = \lambda_1 \ldots (1)$

$2\lambda_1 + \lambda_2 = y \ldots (2)$

$3\lambda_1 + 2\lambda_2 + \lambda_3 = z \ldots (3)$

But (1) in (2)

$2x + \lambda_2 = y \rightarrow \lambda_2 = y - 2x$

$x = \lambda_1, \lambda_2 = y - 2x$ *but in (3)*

$3x + 2(y - 2x) + \lambda_3 = z$

$\rightarrow \lambda_3 = z - 3x - 2(y - 2x) = z - 3x - 2y + 4x$

$\lambda_3 = x + z - 2y$ *then* x_1, x_2, x_3 *generated* R^3

Definition:

Let M *is a proper subspace of a vector space* X *over a field* F*, we say that* M *is a maximal subspace if* N *subspace of* X *such that* $M \subseteq N \subseteq X$ *then* $N = X$*.*

Theorem:

Let M is a proper subspace of a vector space X over a field F then M is a maximal subspace iff $X = [M \cup \{x_0\}]$ for any $x_0 \notin M$.

$\forall x \in X$ is only one way to represent as $x = m + \lambda x_0$ such that $m \in M, \lambda \in F$.

Proof

Since $x_0 \notin M \rightarrow M \subseteq [M \cup \{x_0\}] \subseteq X$

If M is a maximal subspace then by definition we get $[M \cup \{x_0\}] = X$

The converse

Suppose that Nsubspace such that $M \subseteq N \subseteq X$ let $x_0 \in N$, $x_0 \notin M$

$\rightarrow M \subseteq [M \cup \{x_0\}] \subseteq N$

$\therefore X = [M \cup \{x_0\}] \rightarrow X \subseteq N \rightarrow X = N$

Then M is a maximal subspace

Now, we prove that $\forall x \in X$ can be written as only one way $x = m + \lambda x,\ m \in M,\ \lambda \in F$

Since , $X = [M \cup \{x_0\}],\ x \in X$

$\rightarrow x \in [M \cup \{x_0\}] \rightarrow x = m + \lambda x,\ \lambda \in F,\ m \in M$ To prove unique , let $x = m_1 + \lambda_1 x_0$ and $x = m_2 + \lambda_2 x_0$ such that $\lambda_1, \lambda_2 \in F,\ m_1, m_2 \in M$

Since $x = x \rightarrow m_1 + \lambda_1 x_0 = m_2 + \lambda_2 x_0 \ \ m_1 - m_2 = \lambda_2 x_0 - \lambda_1 x_0 \qquad m_1 - m_2 = (\lambda_2 - \lambda_1) x_0$,
since $m_1, m_2 \in M \rightarrow m_1 - m_2 \in M$ *(M subspace)* $\rightarrow$
$(\lambda_2 - \lambda_1) x_0 \in M$ *and*

$$x_0 = \underbrace{\frac{1}{\lambda_2 - \lambda_1}}_{\in F} \underbrace{(m_1 - m_2)}_{\in M}, \ M \text{ subspace}$$

$$\rightarrow \frac{1}{\lambda_2 - \lambda_1}(m_1 - m_2) \in M$$

$\rightarrow x_0 \in M$ *(contradiction) then* $\lambda_1 = \lambda_2$ *and* $m_1 = m_2$

Definition:

Let M_1 *and* M_2 *are tow subspace of a vector space* X *over a field* F *we say that* M_1, M_2 *are disjoint iff* $M_1 \cap M_2 = \{0\}$

Definition:

Let M_1 *and* M_2 *are two subspace of a vector space* X *over a field* F *we say that* X *is direct sum to* M_1 *and* M_2 *and denoted by* $X = M_1 \oplus M_2$*, if for any* $x \in X$ *can be written as unique method* $x = m_1 + m_2$, $m_1 \in M_1, \ m_2 \in M_2$

Theorem:

Let M_1 *and* M_2 *are two subspace of a vector space* X *over a field* F *then Let* M_1 *and* M_2 *are two subspace of a vector space* X *over a field* $X = M_1 \oplus M_2$ *iff*

1- $M_1 \cap M_2 = \{0\}$;
2- $X = M_1 + M_2$

Proof

Suppose that $X = M_1 \bigoplus M_2$ for any $x \in X$ can be written as unique way for sum of elements one of thies element in M_1 and the other in M_2 then $X = M_1 + M_2$

Now, we prove that $M_1 \cap M_2 = \{0\}$

Let $x \in M_1 \cap M_2 \rightarrow x \in M_1 \ \wedge x \in M_2$

If $x \neq 0$, $x \in X$ $\quad x = 0 + x$,
$0 \in M_1, x \in M_2$ or $\quad x = x + 0$, $x \in M_1, 0 \in M_2$

Then x can be written by two method and this cantradection then $M_1 \cap M_2 = \{0\}$

The converse : suppose that (1) and (2) holds
let $x \in X$, since $X = M_1 + M_2$
$\rightarrow x = m_1 + m_2$, $m_1 \in M_1, m_2 \in M_2$

Let $x = x_1 + x_2$, $x_1 \in M_1, x_2 \in M_2$
$x = y_1 + y_2$, $y_1 \in M_1, y_2 \in M_2$
$x = x$

$\rightarrow \ x_1 + x_2 = y_1 + y_2$
$x_1 - y_1 = y_2 - x_2$

Since, $x_1 - y_1 \in M_1$, $y_2 - x_2 \in M_2$ (M_1subspace $x_1, y_1 \in M_1, M_2$ subspace $x_2, y_2 \in M_2$)

$\rightarrow x_1 - y_1, y_2 - x_2 \in M_1$and $x_1 - y_1, y_2 - x_2 \in M_2$

Then $\rightarrow x_1 - y_1, y_2 - x_2 \in M_1 \cap M_2$

But, $M_1 \cap M_2 = \{0\}$

$\rightarrow x_1 - y_1 = \{0\}, \quad y_2 - x_2 = \{0\}$

$\rightarrow x_1 = y_1, \quad y_2 = x_2$

Then we get x *can be written as one way*

Remark:

If the vector space X *direct sum of two subspace* M_1 *and* M_2 *(i.e* $X = M_1 \oplus M_2$ *then* M_2 *is said to be (complement subspace)* M_1 *in* X *(i.e* M_1 *and* M_2 *are said to be complement subspace).*

Theorem:

If M *is a subspace of a vector space* X *over a field* F *then* M *is has a complement subspace in* X

Question:

Show that whether every subspace has unique complement subspace

Example:

Let $X = R^2$ *and*

$M_1 = \{ (x,0), x \in R\}$

$M_2 = \{ (0,x), x \in R\}$

$M_3 = \{ (x,x), x \in R\}$

$X = M_1 \oplus M_2 = M_1 \oplus M_3$ *, every* M_2 *and* M_3 *is a complement of* M_1*.*

Linear independence

Definition:

Let X *be a vector space over afield* X*, a finite set of a vector in* X $\{x_1, x_2, \dots, x_n\}$ *is said to be linear dependent iff , there exists* $\lambda_1, \lambda_2, \dots, \lambda_n \in F$*, such that* $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$

$\lambda_1, \lambda_2, \dots, \lambda_n$ *not all zero*

Other wise, we say that the set is linear independent iff $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$ *then* $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

**If A is a subset of X we say that A is a linear independent if every non empty finite subset of A is a linear independent*

i.e\ if there exist a non – empty finite subset of A is a linear dependent.

Example:

Show that the vector in R^3 is independent or not

1- $x_1 = (1,-2,1),\ x_2 = (2,1,-1),\ \ x_3 = (6,-4,1)$
2- $x_1 = (1,2,-3),\ x_2 = (1,-3,2),\ \ x_3 = (1,-3,2)$

Solution

1- H.W
2- $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$

$$\lambda_1(1,2,-3) + \lambda_2(1,-3,2) + \lambda_3(1,-3,2) = 0$$

$$(\lambda_1, 2\lambda_1, -3\lambda_1) + (\lambda_2, -3\lambda_2, 2\lambda_2) + (\lambda_3, -3\lambda_3, 2\lambda_3) = 0$$

$\lambda_1 + \lambda_2 + \lambda_3 = 0$ . . . *(1)*

$2\lambda_1 + 3\lambda_2 - \lambda_3 = 0$ . . .*(2)*

$-3\lambda_1 + 2\lambda_2 + 5\lambda_3 = 0$ . . . *(3)*

$\lambda_1 + \lambda_2 + \lambda_3 = 0 \quad (\times 2)$

$$\rightarrow 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

$$-2\lambda_1 \mp 3\lambda_2 \underline{+} \lambda_3 = 0$$

$$-\lambda_2 + 3\lambda_3 = 0 \ldots (4)$$

$2\lambda_1 + 3\lambda_2 - \lambda_3 = 0 \ (\times 3) \rightarrow \ 6\lambda_1 + 9\lambda_2 - 3\lambda_3 = 0 \ -3\lambda_1 + 2\lambda_2 + 5\lambda_3 = 0 (\times 2) \rightarrow -6\lambda_1 + 4\lambda_2 + 10\lambda_3 = 0$

$6\lambda_1 + 9\lambda_2 - 3\lambda_3 = 0$

$-6\lambda_1 + 4\lambda_2 + 10\lambda_3 = 0$

$13\lambda_2 + 7\lambda_3 = 0$ *. . . (5)*

From (4) and (5) we get

$-\lambda_2 + 3\lambda_3 = 0 (\times 13) \rightarrow -13\lambda_2 + 39\lambda_3 = 0$

$$13\lambda_2 + 7\lambda_3 = 0$$

$$46\lambda_3 = 0$$

$\rightarrow \ \lambda_3 = 0$ *from (4) or (5)*

We get $\lambda_2 = 0$ *from (1) or (2) or (3) we get*

$$\lambda_1 = 0$$

Then $\lambda_1 = \lambda_2 = \lambda_3 = 0$

Linear independent

Example:

Let x_1, x_2 and x_3 are independent vector space of a vector space X over a field F prove that the vectors $x_1 + x_2, x_1 - x_2, x_1 - 2x_2 + x_3$ are linear independent

Solution

$$\lambda_1(x_1 + x_2) + \lambda_2(x_1 - x_2) + \lambda_3(x_1 - 2x_2 + x_3) = (0,0,0)$$

$$(\lambda_1 x_1 + \lambda_1 x_2) + (\lambda_2 x_1 - \lambda_2 x_2) + (\lambda_3 x_1 - 2\lambda_3 x_2 + \lambda_3 x_3) = (0,0,0)$$

$\lambda_1 x_1 + \lambda_2 x_1 + \lambda_3 x_1 = 0$

$\lambda_1 x_2 - \lambda_2 x_2 - 2\lambda_3 x_2 = 0$

$\lambda_3 x_3 = 0$

$\rightarrow (\lambda_1 + \lambda_2 + \lambda_3)x_1 = 0$

$(\lambda_1 - \lambda_2 - 2\lambda_3)x_2 = 0$

$\lambda_3 x_3 = 0$

Since, x_1, x_2 and x_3 are independent

$\rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 - \lambda_2 - 2\lambda_3 = 0$

$\lambda_3 = 0$

By solution of equation we get

$\lambda_3 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 - \lambda_2 - 2\lambda_3 = 0$

$2\lambda_1 - \lambda_3 = 0$ *by* $\lambda_3 = 0$

Then se get

$2\lambda_1 = 0 \qquad \to \lambda_1 = 0$

And we get $\lambda_2 = 0$

$\to \lambda_1 = \lambda_2 = \lambda_3 = 0$

$\to x_1 + x_2, x_1 - x_2, x_1 - 2x_2 + x_3$ *are linear independent*

Bases and Dimension

<u>Definition:</u>

Let A *be a non- empty subset of a vector space* X *over a field* F*, we say that* A *is a basis for* X *iff* A *is a linear independent and generated* X *,* $X = [A]$

Example:

Let R^n *be a vector space over a field* R*, the set* $\{e_1, e_2, \dots, e_n\}$ *such that* $e_1 = (1,0,\dots,0),\ e_2 = (0,1,\dots,0), e_3 = (0,0,1,\dots,0), . . , e_n = (0,0,\dots,1)$

Is a basis for R^n *and said to be natural basis*

solution

1- $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0$

$\lambda_1(1,0,\dots,0) + \lambda_2(0,1,\dots,0) + \cdots + \lambda_n(0,0,\dots,1) = 0$

$$(\lambda_1, 0, \dots,0) + (0, \lambda_2, \dots,0) + \cdots + (0,0,\dots,\lambda_n) = 0$$

$\lambda_1 = 0$

$\lambda_2 = 0$

$\lambda_3 = 0$

.

.

.

$\lambda_n = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$

Linear independent

2- Let $(x_1, x_2, \dots, x_n) \in R$ *such that*

$$(x_1, x_2, \dots, x_n) = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$$

$\to (x_1, x_2, \dots, x_n) = \lambda_1(1,0,\dots,0) + \lambda_2(0,1,\dots,0) + \cdots + \lambda_n(0,0,\dots,1)$

$\to (x_1, x_2, \dots, x_n) = (\lambda_1, 0, \dots,0) + (0, \lambda_2, \dots,0) + \cdots + (0,0,\dots, \lambda_n)$

$\to x_1 = \lambda_1$

$x_2 = \lambda_2$

.

.

$x_n = \lambda_n$

$\to R^n = [A]$ *, where* $A = \{e_1, e_2, \dots, e_n\}$

$\to$ *generated* R^n

$\to A$ *is a basis*

Remark:

If $X = \{0\}$ *then there is no a subset linear independent of* X *then* X *is a basis.*

Example:

Show that the following vectors is a basis for R^3 or not with prove

a- $x_1 = (1,3,-4),\ x_2 = (1,4,-3), x_3 = (2,3,-1)$

b- $x_1 = (2,4,3),\ x_2 = (0,1,1), x_3 = (0,1,-1)$

Solution:

a- *H.W*

b-

1- $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$

$\lambda_1(2,4,3) + \lambda_2(0,1,1) + \lambda_3(0,1,-1) = 0$

$(2\lambda_1, 4\lambda_1, 3\lambda_1) + (0, \lambda_2, \lambda_2) + (0, \lambda_3, -\lambda_3) = 0$

$2\lambda_1 = 0$ . . . *(1)*

$4\lambda_1 + \lambda_2 + \lambda_3 = 0$ . . . *(2)*

$3\lambda_1 + \lambda_2 - \lambda_3 = 0$ . . . *(3)*

By (1) we get $\lambda_1 = 0$

Cover $\lambda_1 = 0$ *in (2), (3) we get*

$\lambda_2 + \lambda_3 = 0$ . . . *(4)*

$\lambda_2 - \lambda_3 = 0$ . . . *(5)*

$2\lambda_2 = 0 \quad \rightarrow \lambda_2 = 0$

Cover $\lambda_1 = 0$, $\lambda_2 = 0$ *in (2) or (3) we get*

$\lambda_3 = 0 \quad \rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ *linear independent*

2- Let $(a_1, a_2, a_3) \in R^3$ *such that*

$(a_1, a_2, a_3) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$

$\rightarrow\ (a_1, a_2, a_3) = \lambda_1(2,4,3) + \lambda_2(0,1,1) + \lambda_3(0,1,-1)$

$\rightarrow\ (a_1, a_2, a_3) = (2\lambda_1, 4\lambda_1, 3\lambda_1) + (0, \lambda_2, \lambda_2) + (0, \lambda_3, -\lambda_3)$

$a_1 = 2\lambda_1$ *. . . (1)*

$a_2 = 4\lambda_1 + \lambda_2 + \lambda_3$ *. . . (2)*

$a_3 = 3\lambda_1 + \lambda_2 - \lambda_3$ *. . . (3)*

From (1) we get $\lambda_1 = \frac{a_1}{2}$ *cover in (2) and (3)*

$a_2 = 4(\frac{a_1}{2}) + \lambda_2 + \lambda_3$, $a_3 = 3\frac{a_1}{2} + \lambda_2 - \lambda_3$

$a_2 = 2a_1 + \lambda_2 + \lambda_3$ *. . . (4)*

$a_3 = \frac{3}{2}a_1 + \lambda_2 - \lambda_3$ *. . . (5)*

$a_2 + a_3 = 2a_1 + \frac{3}{2}a_1 + 2\lambda_2$

$a_2 + a_3 = \frac{4a_1 + 3a_1}{2} + 2\lambda_2$

$a_2 + a_3 = \frac{7}{2}a_1 + 2\lambda_2$

$2\lambda_2 = a_2 + a_3 - \frac{7}{2}a_1$

$\lambda_2 = \frac{a_2}{2} + \frac{a_3}{2} - \frac{7}{4}a_1$ *cover in (4)*

$2a_1 + \frac{a_2}{2} + \frac{a_3}{2} - \frac{7}{4}a_1 + \lambda_3 = a_2$

$\lambda_3 = a_2 - 2a_1 - \frac{a_2}{2} - \frac{a_3}{2} + \frac{7}{4}a_1$

$= \frac{2-1}{2}a_2 + \frac{7-8}{4}a_1 - \frac{a_3}{2}$

$\lambda_3 = 1a_2 - \frac{1}{4}a_1 - \frac{1}{2}a_3$

→ *Generated* R^3

→ *basis*

Definition:

Let X *be a vector space over a field* F *we say that the number of the element of a basis of* X *is dimension of* X *and denoted by* $dim(X)$

- *Zero vector space* $X = \{0\}$ *othough that denoted not have basis then* $dim(X) = 0$*, we say that the vector space* X *is finite dimension iff* $dim(X) = n, n \in Z^+$ *or* $n = 0$ *we say that the vector space* X *is an infinite dimension is the number of the element of a basis of* X *is infinite i.e*$dim(X) = \infty$ *.*

Example:

Find the dimension of he vector space R^3 *over a field* R *where* $S = \{ (1,0,0), (0,1,0), (0,0,1)\}$ *is a basis*

Solution

$dim(R^3) = 3$

Convexity

Definition:

Let A be a subset of a vector space X over a field F A is said to be convex set iff $\lambda x + (1-\lambda)y \in A \quad \forall x, y \in A, \lambda \in F, 0 \le \lambda \le 1$

OR

A is a convex set iff $\lambda A + (1-\lambda)A \subseteq A \quad \forall \lambda \in F, 0 \le \lambda \le 1$

- *The empty set $\emptyset$, and the set contain only one element are convex set*

Example:

Every subspace is a convex set but the conversely need not true in general

Solution

Let M be a subspace of a vector space X then $\forall x, y \in M, \alpha, \beta \in F$

$$\rightarrow \alpha x + \beta y \in M$$

Put $\lambda = \alpha$ *,* $1 - \lambda = \beta$ *,* $\quad 0 \le \lambda \le 1$

$$\rightarrow \lambda x + (1 - \lambda) y \in M, \qquad \forall x, y \in M, \lambda \in F$$

$\rightarrow M$ *is a convex set*

The conversely (H.W) by example

Theorem:

If A, B *are convex sets of a vector space over a field then ,* $A + B$ *is a convex set of a vector space.*

Proof

Let $x, y \in A + B, \lambda \in F, 0 \le \lambda \le 1$

Since $x \in A + B \rightarrow \quad x = a + b \quad \ni a \in A, b \in B$ *and* $y \in A + B \rightarrow \; y = c + d \quad \ni c \in A, d \in B$

Since Since A *is a convex set and* $a, c \in A$

$$\rightarrow \lambda a + (1 - \lambda) c \in A, \qquad \lambda \in F, 0 \le \lambda \le 1$$

And since B *is a convex set and* $b, d \in B$

$$\rightarrow \lambda b + (1 - \lambda) d \in B, \qquad \lambda \in F, 0 \le \lambda \le 1$$

Now,

$$\begin{aligned} \rightarrow \lambda x + (1 - \lambda) y &= \lambda(a + b) + (1 - \lambda)(c + d) \\ &= (\lambda a + \lambda b) + (1 - \lambda) c + (1 - \lambda) d \\ &= \underbrace{\lambda a + (1 - \lambda) b}_{\in A} + \underbrace{\lambda c + (1 - \lambda) d}_{\in B} \in A + B, \\ & \lambda \in F, 0 \le \lambda \le 1 \end{aligned}$$

Then $\lambda x + (1-\lambda)y \in A+B,\ \lambda \in F, 0 \le \lambda \le 1$

$A+B$ *is a convex set*

Remark:

If A *is a subset of a vector space* X *then* $(\alpha+\beta)A \subseteq \alpha A + \beta A$

Proof

Let $x \in (\alpha+\beta)A$

$\rightarrow x = (\alpha+\beta)a,\ a \in A$

$\quad = \alpha a + \beta a\ \in \alpha A + \beta A$

$\rightarrow x \in \alpha A + \beta A$

$\rightarrow (\alpha+\beta)A \subseteq \alpha A + \beta A$

But $\alpha A + \beta A \nsubseteq (\alpha+\beta)A$

Theorem:

If A *be a subset of a vector space* X *over* F *then* A *is a convex set iff*

$(\alpha+\beta)A = \alpha A + \beta A, \alpha, \beta \in R^+$

Proof

Let A *is a convex set ,*

we prove that $(\alpha+\beta)A = \alpha A + \beta A$

since by above remark $(\alpha+\beta)A \subseteq \alpha A+\beta A...$*(1)*

It remains to show that $\alpha A+\beta A \subseteq (\alpha+\beta)A$

Let $x \in \alpha A+\beta A$

$\rightarrow x=\alpha a+\beta b,\ \ \alpha,\beta \in R^{+}, a,b \in A$

$x=(\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}a,\frac{\beta}{\alpha+\beta}b\right)$

Put, $\lambda=\frac{\alpha}{\alpha+\beta},\ 1-\lambda=\frac{\beta}{\alpha+\beta}$

Since, $\alpha,\beta \in R^{+},\ \lambda \geq 0$ *and since*

$\alpha \leq \alpha+\beta \ \ \rightarrow \lambda \leq 1$

$\rightarrow 0 \leq \lambda \leq 1$

And since, A *is a convex set* $\rightarrow \lambda a+(1-\lambda)b \in A$

i.e $\frac{\alpha}{\alpha+\beta}a,\frac{\beta}{\alpha+\beta}b \in A$

then $x \in (\alpha+\beta)A$

$\rightarrow \alpha A+\beta A \subseteq (\alpha+\beta)A...$ *(2)*

From (1) and (2) we get $(\alpha+\beta)A=\alpha A+\beta A$

<u>The converse</u>

Let $(\alpha+\beta)A=\alpha A+\beta A,\ \ \alpha,\beta R^{+}$

Let $\lambda \in F, 0 \leq \lambda \leq 1 \ \ \rightarrow 1-\lambda \geq 0$

Then we get

$$\lambda A+(1-\lambda)A=\big(\lambda+(1-\lambda)\big)A=A \subseteq A$$

Then , $\lambda A+(1-\lambda)A \subseteq A$

Then A is a convex set

Theorem:

Let A, B are two convex sets of a vector space over a field then $A \cap B$ is a convex set

Proof

Let $x, y \in A \cap B,\ \lambda \in F,\ 0 \leq \lambda \leq 1$

Since, $x, y \in A \cap B \ \rightarrow x, y \in A \ \wedge \ x, y \in B$

Now, $x, y \in A,\ A$ convex set , $\lambda \in F,\ 0 \leq \lambda \leq 1$

$\rightarrow \lambda x + (1-\lambda)y \in A$

Also, $x, y \in B, B$ convex set $\lambda \in F,\ 0 \leq \lambda \leq 1$

$\rightarrow \lambda x + (1-\lambda)y \in B$

Now, $\lambda x + (1-\lambda)y \in A \wedge \lambda x + (1-\lambda)y \in B$

$$\rightarrow \lambda x + (1-\lambda)y \in A \cap B, \lambda \in F, \qquad 0 \leq \lambda \leq 1$$

Then $A \cap B$ is a convex set

Definition:

Let A be a subset of a vector space X then the intersection of all convex subset of X containing A is the smallest convex subset of X containing A is called the convex hull of A and denoted by $conv(A)$,

$$conv(A) = \{A_i, A \text{ } convex \text{ } set \text{ } and \text{ } A \subseteq A_i\}$$

Remark:

1- $A \subseteq conv(A)$
2- A is a convex set iff $A = conv(A)$

Definition:

Let X be a vector space over a field F let $x_1, x_2, \dots, x_n \in X$, we say that the vector $x \in X$ is a convex combination for the vector $x_1, x_2, \dots, x_n$ if $x = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$

Affine sets

Definition:

Let A be a subset of a vector space X over a field F A is said to be affine set iff $\lambda x + (1-\lambda) y \in A$ $\forall x, y \in A, \lambda \in F$

- The empty set $\emptyset$, and the set contain only one element are affine sets

Example:

Every affine set is a convex set but the conversely need not true in general

Solution

Let A be affine set in X then

$\forall x, y \in A, \lambda \in F$

$\rightarrow \lambda x + (1 - \lambda) y \in A$

For any $\lambda \in F$

$\rightarrow \lambda x + (1 - \lambda) y \in A \; x, y \in A, \lambda \in F, 0 \leq \lambda \leq 1$

$\rightarrow$ *A is a convex set*

The conversely (H.W) by example

Example:

Every subspace is affine set but the conversely need not true in general

Theorem:

If A, B are affine sets of a vector space over a field then , $A + B$ is affine set of a vector space.

Proof

Let $x, y \in A + B, \lambda \in F,$

Since $x \in A + B \rightarrow \quad x = a + b \quad \ni a \in A, b \in B$ *and* $y \in A + B \rightarrow \ y = c + d \quad \ni c \in A, d \in B$

Since A *is a convex set and* $a, c \in A$

$$\rightarrow \lambda a + (1 - \lambda)c \in A, \qquad \lambda \in F,$$

And since B *is a convex set and* $b, d \in B$

$$\rightarrow \lambda b + (1 - \lambda)d \in B, \qquad \lambda \in F,$$

Now,

$$\begin{aligned} \rightarrow \lambda x + (1 - \lambda)y &= \lambda(a + b) + (1 - \lambda)(c + d) \\ &= (\lambda a + \lambda b) + (1 - \lambda)c + (1 - \lambda)d \\ &= \underbrace{\lambda a + (1 - \lambda)b}_{\in A} + \underbrace{\lambda c + (1 - \lambda)d}_{\in B} \in A + B, \\ &\lambda \in F, \end{aligned}$$

Then $\lambda x + (1 - \lambda)y \in A + B, \ \lambda \in F$

$A + B$ *is affine set*

Theorem:

Let X *be a vector space and let* $x_0 \in X$ *then*

1- *If* M *is a subspace in* X *then* $x_0 + M$ *is affine set in* X
2- *If* A *is affine set in* X *then* $A - x_0$*is a subspace in* X

Proof

1- *Suppose that* M *is a subspace and let* $x, y \in x_0 + M, \lambda \in F$

$x = x_0 + m_1, y = x_0 + m_2, m_1, m_2 \in M$

$$\lambda x + (1-\lambda)y = \lambda(x_0 + m_1) + (1-\lambda)(x_0 + m_2)$$

$$= \lambda x_0 + \lambda m_1 + (1-\lambda)x_0 + (1-\lambda)m_2$$

$$= \lambda x_0 + (1-\lambda)x_0 + \lambda m_1 + (1-\lambda)m_2$$

$$= (\lambda x_0 + x_0 - \lambda x_0) + \lambda m_1 + (1-\lambda)m_2$$

$$= x_0 + \lambda m_1 + (1-\lambda)m_2$$

Since, $m_1, m_2 \in M, M$ *is a subspace*

$\rightarrow \underbrace{\lambda}_{\alpha} m_1 + \underbrace{(1-\lambda)}_{\beta} m_2 \in M$

Then $x_0 + \lambda m_1 + (1-\lambda)m_2 \in x_0 + M$

$\rightarrow \lambda x + (1-\lambda)y \in x_0 + M$

$\rightarrow x_0 + M$ *is affine set*

2- H.W

Theorem: H.W

Let A, B *are two affine sets of a vector space over a field then* $A \cap B$ *is affine set*

Definition:

Let A *be a subset of a vector space* X *the the smallest set* X *contain* A *is called affine set generated by* A *and denoted by* $aff(A)$ *,*

$aff(A) = \{A_i, A\ caffine\ set\ and\ A \subseteq A_i\}$

Remark:

1- $x = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in A, \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = A$
2- $A$is affine set iff $A = aff(A)$

Linear transformation

Definition:

Let X, Y are vector space over a field F, the function $f: X \to Y$ is said to be linear transformation if the conditions holds

1- $f(x + y) = f(x) + f(y), \forall x, y \in X$
2- $f(\lambda x) = \lambda f(x), \forall x \in X, \lambda \in F$

This two conditions equivalent to the condition $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \forall x, y \in X, \alpha, \beta \in F$

**- linear transformation $f: X \to Y$ is said to be linear function on X*

Example:

Let $f: R^3 \rightarrow R^2$ *define as follows*

$$f(x_1, x_2, x_3) = (x_1, x_2)$$

$\forall (x_1, x_2, x_3) \in R^3$ *show that* f *linear transformation*

Solution

Let $x, y \in R^3, \alpha, \beta \in F$

Since $x \in R^3 \rightarrow x = (x_1, x_2, x_3)$ *and*

$y \in R^3 \rightarrow x = (y_1, y_2, y_3)$

$$\alpha x + \beta y = \alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)$$

$$= (\alpha x_1, \alpha x_2, \alpha x_3) + (\beta y_1, \beta y_2, \beta y_3)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$= (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$$

$$= \alpha(x_1, x_2) + \beta(y_1, y_2)$$

$$= \alpha f(x) + \beta f(y)$$

Then f *is a linear transformation*

Example:

Show that the function $f: R^2 \rightarrow R$ *define as follows*

$$f(x_1, x_2) = x_1 x_2, \forall (x_1, x_2) \in R^2$$

Is not linear transformation

Solution

Let $x, y \in R^2$ *such that*

$x = (x_1, x_2), y = (y_1, y_2)$

$$x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$f(x + y) = f(x_1 + y_1, x_2 + y_2) = (x_1 + y_1)(x_2 + y_2)$$
$$= x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2 \dots (1)$$

$$f(x) + f(y) = f(x_1, x_2) + f(y_1, y_2) = x_1 x_2 + y_1 y_2 \dots (2)$$

Since (1)≠(2)

$$\rightarrow f(x + y) \neq f(x) + f(y)$$

f *is not linear transformation*

Remark:

1- *Zero transformation: if* X *and* Y *are two vector spaces over a field* F *then, the function* $f: X \rightarrow Y$ *define as follows ,* $f(x) = 0, \forall x \in X$ *is a linear transformation*
2- *Identity transformation: if* X *and* Y *are two vector spaces over a field* F *then, the function* $f: X \rightarrow Y$ *define as follows ,* $f(x) = x, \forall x \in X$ *is a linear transformation*

Some properties of linear transformation

Theorem:

If $f: X \to Y$ *linear transformation then*

1- $f(0) = 0$
2- $f(-x) = -f(x)$
3- $f(x - y) = f(x) - f(y)$
4- $f(\sum_{i=1}^{n} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i f(x_i)$

Proof

1- *Since* $0 = 0 \cdot 0$

$f(0) = f(0 \cdot 0) = 0 \cdot f(0) = 0$

2- $f(-x) = f\big((-1)x\big) = -1f(x) = -f(x), -1 \in F$
3- *H.W*
4- *We prove by mathematical induction*

If $n = 1$ *then*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = f(\lambda_1 x_1) = \lambda_1 f(x_1) = \sum_{i=1}^{n} \lambda_i f(x_i)$$

Let the statement is true when $n = k$

i.e $f\big(\sum_{i=1}^{k} \lambda_i x_i\big) = \sum_{i=1}^{k} \lambda_i f(x_i)$

we prove that the statement is a true for $n = k + 1$

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left(\sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1}\right)$$

$$= \sum_{i=1}^{k} \lambda_i\, f(x_i) + \lambda_{k+1} f(x_{k+1}) = \sum_{i=1}^{k+1} \lambda_i\, f(x_i)$$

Then the statement is true for any n *when* n *is a positive integer number*

Theorem:

If X, Y *and* Z *are vector space over afield* F *and* $f: X \to Y, g: Y \to Z$ *are linear transformation then* $g \circ f: X \to Z$ *is a linear transformation*

Proof

Let $x, y \in X, \alpha, \beta \in F$

$$g \circ f(\alpha x + \beta y)$$

$$= g\big(f(\alpha x + \beta y)\big)$$

$$= g\big(\alpha f(x) + \beta f(y)\big)$$

$$= \alpha g\big(f(x)\big) + \beta g\big(f(y)\big)$$

$$= \alpha g \circ f(x) + \beta g \circ f(y)$$

$g \circ f$ *is a linear transformation*

Theorem:

If $f: X \to Y$ *is a linear transformation and* $A \subseteq X, B \subseteq Y$ *then*

1- *If* A *is a subspace (convex set) in* X *then* $f(A)$ *is a subspace (convex set) in* Y

2- *If B is a subspace (convex set) in Y then $f^{-1}(B)$ is a subspace (convex set) in X*

Proof

1- *Let A be a subspace in X*

We prove that $f(A)$ is a subspace in Y

$$f(A) = \{f(x), x \in A\} \subseteq Y$$

Since A is a subspace $\rightarrow 0 \in A$

$0 = f(0) \in f(A)$

$\rightarrow f(A) \neq \emptyset$

Let $\alpha, \beta \in F, a, b \in f(A)$

$$a = f(c), b = f(d), c, d \in A$$

$$\alpha a + \beta b = \alpha f(c) + \beta f(d)$$

$$= f(\alpha c) + f(\beta d)$$

$$= f(\alpha c + \beta d)$$

Since $c, d \in A, A$ subspace $\alpha, \beta \in F$

$\rightarrow \alpha c + \beta d \in A$

$\rightarrow f(\alpha c + \beta d) \in f(A)$

$\rightarrow \alpha a + \beta b \in f(A)$

$\rightarrow f(A)$ *is a subspace*

Let B is a convex set in Y

We prove that $f^{-1}(B)$ is a convex set in X

$$f^{-1}(B) = \{x \in X, f(x) \in B\} \subseteq X$$

Let $a, b \in f^{-1}(B), \gamma \in F, 0 \leq \gamma \leq 1$

$$f(a), f(b) \in B$$

Since B *convex set*

$\rightarrow \gamma f(a) + (1-\gamma)f(b) \in B$

$$\left.\begin{array}{l} \rightarrow f(\gamma a) + f\big((1-\gamma)b\big) \in B \\ \rightarrow f(\gamma a + (1-\gamma)b) \in B \end{array}\right\} \text{linear transf}$$

$\rightarrow \gamma a + (1-\gamma)b \in f^{-1}(B)$

$f^{-1}(B)$ *is a convex set*

Remark:

If X *is a vector space of finite dimension and let* $\{x_1, x_2, \dots, x_n\} = B$ *is a basis of* X *then* $x_i \in X$ *can be written unique method as*

$$x = \sum_{i=1}^{n} \gamma_i x_i, \gamma_i \in F$$

Theorem:

If X *be a vector space and* $\{x_1, x_2, \dots, x_n\}$ *is a basis of* X *then, foe any set* $\{y_1, y_2, \dots, y_n\}$ *contain* n *vectors in voctor space* Y *there exists only one linear transformation method*

$f: X \to Y$, such that $f(x_i) = y_i$

Proof

Let $x \in X$, since $\{x_1, x_2, \dots, x_n\}$ basis

$\to x$ has unique method

$x = \sum_{i=1}^{n} \gamma_i x_i \, , \gamma_i \in F$

Define the function $f: X \to Y$, such that $f(x) = \sum_{i=1}^{n} \gamma_i \, y_i$ to prove that

1- *f linear transformation*

Let $x, yX, \alpha, \in F$

Then $x = \sum_{i=1}^{n} \delta_i x_i \, , y = \sum_{i=1}^{n} \mu_i x_i$

$$\to \alpha x + \beta y = \alpha \sum_{i=1}^{n} \delta_i x_i + \beta \sum_{i=1}^{n} \mu_i x_i$$

$$= \sum_{i=1}^{n} (\alpha \delta_i + \beta \mu_i) x_i$$

$$f(\alpha x + \beta y) = f\left(\sum_{i=1}^{n} (\alpha \delta_i + \beta \mu_i) x_i\right) = \sum_{i=1}^{n} (\alpha \delta_i + \beta \mu_i) y_i$$

$$= \alpha \sum_{i=1}^{n} \delta_i y_i + \beta \sum_{i=1}^{n} \mu_i y_i$$

$$= \alpha f(x) + \beta f(y)$$

2- *$f(x_i) = y_i$*

$$x_i = 0 \cdot x_1 + 0 \cdot x_2 = \cdots + 1 \cdot x_i + \cdots + 0 \cdot x_n$$

$f(x_i)$

$$= f(0 \cdot x_1 + 0 \cdot x_2 + \cdots + 1 \cdot x_i + \cdots + 0 \cdot x_n)$$

$$= f(0 \cdot x_1) + f(0 \cdot x_2) + \cdots + f(1 \cdot x_i) + \cdots + f(0 \cdot x_n)$$

$$= 0 \cdot f(x_1) + 0 \cdot f(x_2) + \cdots + 1 \cdot f(x_i) + \cdots + 0 \cdot f(x_n)$$

$$= 0 + 0 + \cdots + 1 \cdot y_i + \cdots + 0 = y_i$$

$$f(x_i) = y_i$$

3- f has only one linear transformation (unique method)

If $g: X \to Y$ is a linear transformation such that $g(x_i) = y_i$

To prove that $f(x) = g(x)$

Let $x \in X \to x$ has unique method

$$x = \sum_{i=1}^{n} \gamma_i x_i$$

$$g(x) = g\left(\sum_{i=1}^{n} \gamma_i x_i\right) = \sum_{i=1}^{n} \gamma_i \, g(x_i) = \sum_{i=1}^{n} \gamma_i y_i = f(x)$$

Example:

Find the linear transformation $f: R^2 \to R^2$ such that

$$f(1,1) = (0,2), f(3,1) = (2,-4)$$

Solution

$Let\ (x, y) \in R^2$

$$\rightarrow (x, y) = \alpha_1(3,1) + \alpha_2(1,1)$$

$$= (3\alpha_1, \alpha_1) + (\alpha_2, \alpha_2)$$

$$= (3\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$$

$\rightarrow x = 3\alpha_1 + \alpha_2$

$-y = -\alpha_1 \mp \alpha_2$

$x - y = 3\alpha_1 - \alpha_1$

$x - y = 2\alpha_1$

$\alpha_1 = \frac{x-y}{2}$

$\rightarrow x = 3\alpha_1 + \alpha_2$

$y = \alpha_1 + \alpha_2 \quad * 3$

$x = 3\alpha_1 + \alpha_2$

$-3y = -3\alpha_1 \mp 3\alpha_2$

$x - 3y = \alpha_2 - 3\alpha_2$

$x - 3y = -2\alpha_2$

$\alpha_2 = \frac{3y-x}{2}$

$f(x, y) = \alpha_1 f(3,1) + \alpha_2 f(1,1)$

$= \alpha_1(2, -4) + \alpha_2(0,2)$

$$= \frac{x-y}{2}(2, -4) + \frac{3y-x}{2}(0,2)$$

$$= (x - y, 2y - 2x) + (0,3y - x)$$
$$= (x - y, 5y - 3x)$$

Definition:

Let $f: X \to Y$ is a linear transformation then the kernel of f denoted by ker(f) and define as follows:

$ker(f) = \{x \in X: f(x) = 0\}$

i.e\ ker(f) $= f^{-1}(\{0\})$

Theorem:

if $f: X \to Y$ is a linear transformation then

1- Ker(f) is a subspace of X
2- Ker(f) $= \{0\}$ iff f is one- to -one

Proof

1-
a- Since $f(0) = 0$

$\to 0 \in ker(f)$

$\to ker(f) \neq \emptyset$

b- Let $x, y \in ker(f)$ *,* $\alpha, \beta \in F$

Since $x \in ker(f) \to f(x) = 0$ *and*

$y \in ker(f) \to f(y) = 0$

Now, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = 0 + 0 = 0$

Then $f(\alpha x + \beta y) = 0$

$\rightarrow \alpha x + \beta y \in ker(f)$

$\rightarrow ker(f)$ *is a subspace*

2- *Suppose that ker(*f*)=* $\{0\}$

To prove that f *is one –to – one*

Let $x, y \in X$ *such that* $f(x) = f(y)$

$f(x) - f(y) = 0$

$\rightarrow f(x - y) = 0$

$\rightarrow x - y \in ker(f)$

$x - y = 0$

$\rightarrow x = y$

$\rightarrow f$ *is one-to-one*

<u>The converse :</u>

Suppose that f *is linear transformation one –to- one*

To show that $ker(f) = \{0\}$

Let $x \in ker(f) \rightarrow f(x) = 0$

Since $f(0) = 0$

$0 = 0$

$\rightarrow f(x) = f(0)$

Since f *linear one-to-one*

$\rightarrow x = 0$

$\rightarrow ker(f) = \{0\}$

Theorem:

Let $f: X \rightarrow Y$ *linear transformation , if* X *finite dimension then the subspace* $f(x)$ *also finite dimension*

Proof

If $dim(X) = 0$ *then* $X = \{0\}$

And $f(X) = \{f(x): x \in X\} = \{0\}$

$\rightarrow dim\big(f(x)\big) = 0$

If $dim(X) = n > 0$

Let $B = \{x_1, x_2, \dots, x_n\}$ *basis of* X *and let* $y \in f(X)$

$\rightarrow \exists x \in X$ *such that* $f(x) = y$

Since, B *basis of* $X \rightarrow x$ *has only one method*

$$x = \sum_{i=1}^{n} \lambda_i x_i \, , \lambda_i \in F$$

And

$$f(x) = \sum_{i=1}^{n} \lambda_i f(x_i)$$

$$y = \sum_{i=1}^{n} \lambda_i f(x_i) \ \dots (1)$$

Let $B' = \{f(x_1), f(x_2), \dots f(x_n)\}$

$\rightarrow B'$ *basis of* $f(x)$

$\rightarrow f(X)$ *finite dimension*

Sylvester's law

Let $f: X \rightarrow Y$ *linear transformation , if* X *finite dimension then*

$$dim(X) = dim(ker(f)) + dim\big(f(x)\big)$$

Corollary:

Let $f: X \rightarrow Y$ *linear transformation if* X, Y *are finite dimension such that* $dim(X) = dim(Y)$ *then the transformation* f *is a one- to one iff is one to*

Proof

Suppose that f *is a linear transformation* $dim(X) = dim(Y)$ *and* f *is one- to –one*

Since f *is one-to-one*

$\rightarrow ker(f) = \{0\}$ *(by theorem)*

$\rightarrow dim(ker(f)) = 0$

By sylvester's law

$$dim(X) = dim(ker(f)) + dim\big(f(x)\big)$$

$$\rightarrow dim(X) = dim\big(f(x)\big)$$

Then $dim\big(f(x)\big) = dim(y)$

$$\rightarrow f(x) = y$$

$\rightarrow f$ *is one to*

The converse

Suppose that f *is one to*

$\rightarrow y = f(x)$ *and*

$dim(X) = dim(y) = dimf\big(f(x)\big)$

$$\rightarrow dim\big(f(x)\big) = 0 \rightarrow ker(f) = 0$$

$\rightarrow f$ *one-to-one*

Definition:

Let X, Y *vector space over afield* F*, we say that* X, Y *linear isomorphic and write* $X \approx Y$ *if linear transformation (one-to-one, one to)*

From X *to* Y *the function is called linear isomorphism*

Theorem:

If X *be a vector space over* F *and* X *finite dimension* $dim(X) = n$ *then* $X \approx F^n$

Proof

Let $\{x_1, x_2, \dots, x_n\}$ *basis of* X

$$\rightarrow \forall x \in X, x = \sum_{i=1}^{n} \lambda_i x_i \,, \lambda_i \in F$$

Define the function
$f: X \to F^n$ *as follows*

$$f(x) = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

We prove that f *is linear transformation , one-to-one, one to H.W*

<u>*Theorem:*</u>

If X, Y *are two finite dimension vectors space over a field* F *then* $X \approx Y$ *iff* $dim(X) = dim(Y)$

Proof

Let $X \approx Y$

$\to \exists$ *linear transformation* $f: X \to Y$

$\to ker(f) = 0, f(x) = y$

$$dim(X) = dim(ker(f)) + dim\big(f(x)\big)$$

$$= 0 + dim(y)$$

$$= dim(y)$$

<u>*The converse*</u>

By theorem

$$X \approx F^n$$

And $y = F^n$

Then $X \approx Y$

Metric And Normed Space

Definition:

Let X *be a non empty set a function* $d: X \times X \to R$ *such that*

1- $d(x,y) \geq 0, \forall x,y \in X$
2- $d(x,y) = 0$ *iff* $x = y, \forall x,y \in X$
3- $d(x,y) = d(y,x), \forall x,y \in X$
4- $d(x,y) \leq d(x,z) = d(z,y), \forall x,y,z \in X$

Then (X,d) *is said to be metric space*

Definition:

Let X *be a vector space over a a field* R *then a norm on* X *is a map* $\|.\|: X \to R$ *such that*

1- $\|x\| \geq 0, \forall x \in X$
2- $\|x\| = 0$ *iff* $x = 0$
3- $\|\lambda x\| = |\lambda|\|x\|, \forall x \in X, \lambda \in R$
4- $\|x + y\| \leq \|x\| + \|y\|, \forall x,y \in X$

Then $(X,\|.\|)$ *is said to be normed space*

Example:

Let $X = R$ *be a vector space over* R *and* $\|.\|: R \to R$ *such that* $\|x\| = |x|, \forall x \in R$ *show that* $\|.\|$*is a norm on* R

Solution

1- *Let* $x \in R \to \|x\| = |x| \geq 0 \to \|x\| \geq 0$
2- *Let* $x \in R$ *if* $\|x\| = 0 \to |x| = 0$
$\to x = 0$ *if* $x = 0 \to |x| = 0 \to \|x\| = |x| = 0 \to \|x\| = 0$
3- *Let* $x \in R$ *and* $\lambda \in R$ $\|\lambda x\| = |\lambda x| = |\lambda||x| = |\lambda|\|x\|$
4- *Let* $x, y \in R$

$$\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$$

$$\therefore \ \|x + y\| \leq \|x\| + \|y\|$$

$\to \|.\|$ *norm on* R

$\to (X, \|x\|)$ *normed space*

Remark:

1- *Holder's inquailty for* $p, q \in R, \ni \frac{1}{p} + \frac{1}{q} = 1$ *then*
$\sum|x_i y_i| \leq (\sum|x_i|^p)^{\frac{1}{p}} (\sum|y_i|^p)^{\frac{1}{q}}$ *if* $p = 2, q = 2$ *then*
$\sum|x_i y_i| \leq (\sum|x_i|^2)^{\frac{1}{2}} (\sum|y_i|^2)^{\frac{1}{2}}$ *is called cushy – schwar's inequality*
2- *Minkowk's inequality*

If $p \geq 1$ *then*

$$\left(\sum |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}}$$

Example:

Let $X = R^n$*, be a vector space over* R *and* $\|.\|_1, \|.\|_2, \|.\|_3$ *and* $\|.\|_4 : R^n \to R$ *define as follows.*

$\forall x \in R^n, x = (x_1, x_2, \dots, x_n)$

1-$\|.\|_1 = (\sum_{i=1}^{n} x_i^{\,2})^{\frac{1}{2}}$

2-$\|.\|_2 = \sum_{i=1}^{n} |x_i|$

3-$\|.\|_3 = max\{|x_1|, |x_2|, \dots, |x_n|\}$

4-$\|.\|_4 = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$

show that $\|.\|_i$ *are normed space on* R^n *or not*

Solution

$\|.\|_1$

1- *Since* $x_i^{\,2} \geq 0, \forall i = 0,1,2, \dots, n \to \sum x_i^{\,2} \geq 0 \to (\sum x_i^{\,2})^{\frac{1}{2}} \geq 0$ *then* $\|x\| \geq 0$

2- *Let* $\|x\| = 0$

$$\leftrightarrow \left(\sum x_i^{\,2}\right)^{\frac{1}{2}} = 0$$

$$\leftrightarrow \sum x_i^{\,2} = 0$$

$$\leftrightarrow x_i^{\,2} = 0$$

$$\leftrightarrow x_i = 0$$

$$\leftrightarrow x = 0$$

3- Let $x \in R^n, \lambda \in R$

$$\lambda x = \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

$$\|\lambda x\| = \left(\sum_{i=1}^{n} (\lambda x_i)^2\right)^{\frac{1}{2}} = |\lambda| \left(\sum_{i=1}^{n} {x_i}^2\right)^{\frac{1}{2}} = |\lambda| \|x\|$$

4- Let $x, y \in R^n$

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$\|x + y\| = \left(\sum_{i=1}^{n} (x_i + y_i)^2\right)^{\frac{1}{2}}$$

By using minkowsk's inquelity

$$\leq \left(\sum_{i=1}^{n} {x_i}^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} {y_i}^2\right)^{\frac{1}{2}}$$

$$= \|x\| + \|y\|$$

$$\to \|x + y\| \leq \|x\| + \|y\|$$

$\|.\|_2$

1- Since $|x_i| \geq 0, \forall i = 0,1,2, \dots, n$

$\rightarrow \sum |x_i| \geq 0 \rightarrow \|x\| \geq 0$

2- Let $\|x\| = 0$

$$\leftrightarrow \sum_{i=1}^{n} |x_i| = 0$$

$$\leftrightarrow |x_i| = 0$$

$$\leftrightarrow x_i = 0$$

$$\leftrightarrow x = 0$$

3- Let $x \in R^n, \lambda \in R$

$$\lambda x = \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

$$\|\lambda x\| = \sum_{i=1}^{n} |\lambda x_i| = \sum_{i=1}^{n} |\lambda||x_i| = |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda|\|x\|$$

4- Let $x, y \in R^n$

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$\|x + y\| = \sum_{i=1}^{n} |x_i + y_i|$$
$$\leq \sum_{i=1}^{n} (|x_i| + |y_i|)$$
$$\leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\| + \|y\|$$

$$\rightarrow \|x + y\| \leq \|x\| + \|y\|$$

$\|.\|_3$

1- Since $|x_i| \geq 0, \forall i = 0,1,2, \dots, n$

$\rightarrow max\{|x_1|, |x_2|, \dots, |x_n|\} \geq 0 \rightarrow \|x\| \geq 0$

2- Let $\|x\| = 0$

$$\leftrightarrow max\{|x_1|, |x_2|, \dots, |x_n|\} = 0$$

$$\leftrightarrow |x_i| = 0$$

$$\leftrightarrow x_i = 0$$

$$\leftrightarrow x = 0$$

3- Let $x \in R^n, \lambda \in R$

$$\lambda x = \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

$$\|\lambda x\| = max\{|\lambda x_1|, |\lambda x_2|, \dots, |\lambda x_n|\}$$
$$= |\lambda| max\{|x_1|, |x_2|, \dots, |x_n|\} = |\lambda|\|x\|$$

4- Let $x, y \in R^n$

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$\|x + y\|$

$= max\{|x_1 + y_1|, |x_2 + y_2|, \dots, |x_n + y_n|\} \leq max\{|x_1| + |y_1|, |x_2| + |y_2|, \dots, |x_n| + |y_n|\}$

$$\leq max\{|x_1|, |x_2|, \dots, |x_n|\} +$$

$$max\{|y_1|, |y_2|, \dots, |y_n|\}$$

$$= \|x\| + \|y\|$$

$\rightarrow \|x+y\| \leq \|x\|+\|y\|$

$\|.\|_4$ *H.W*

Remark:

Every subspace of normed space is a normed space

Theorem:

Let X be a normed space then

1- $\|0\|=0$
2- $\|-x\|=\|x\|, \forall x \in X$
3- $\|x-y\|=\|y-x\|, \forall x, y \in X$
4- $|\|x\|+\|y\|| \leq \|x-y\| \forall x, y \in X$

Proof

1-,2-,3- H.W

4-$x=(x-y)+y$

$$\|x\|=\|(x-y)+y\| \leq \|x-y\|+\|y\|$$

$\rightarrow \|x\|-\|y\| \leq \|x-y\|$*…(1)*

$y=(y-x)+x$

$$\|y\|=\|(y-x)+x\| \leq \|y-x\|+\|x\|$$

$$\rightarrow \|y\|-\|x\| \leq \|y-x\|=\|x-y\|$$

$$\rightarrow \|y\|-\|x\| \leq \|x-y\| \quad (*-1)$$

$$-(\|y\|-\|x\|) \geq -\|x-y\|$$

$\rightarrow \|x\|-\|y\| \geq -\|x-y\|$*…(2)*

From (1) and (2) we get

$-\|x-y\| \leq \|x\|-\|y\| \leq \|x-y\|$

$$\rightarrow |\|x\|+\|y\|| \ \leq \|x-y\|$$

Example:

Let $1 \leq P \leq \infty$ *and* $\|.\|: \ell^P \rightarrow R$ *such that* $x \in \ell^P, x = (x_1, x_2, \dots, x_n, \dots)$ *then* $\|.\|$ *is a normed on* ℓ^P *where* $\|.\| = (\sum |x_i|^p)^{\frac{1}{p}}$

Solution

1- Let $x \in \ell^P \rightarrow x = (x_1, x_2, \dots, x_n, \dots)$ $\|x\| = (\sum |x_i|^p)^{\frac{1}{p}}$

$\therefore |x_i| \geq 0, \forall i \rightarrow |x_i|^p \geq 0, \forall i$

$\rightarrow \sum |x_i|^p \geq 0 \rightarrow (\sum |x_i|^p)^{\frac{1}{p}} \geq 0$ *then* $\|x\| \geq 0$

2- Let $x \in \ell^p$, $x = (x_1, x_2, \dots, x_n, \dots)$

if $\|x\| = 0 \rightarrow (\sum |x_i|^p)^{\frac{1}{p}} = 0$

$$\rightarrow \sum |x_i|^p = 0 \rightarrow |x_i|^p = 0 \rightarrow |x_i| = 0$$

$$\rightarrow x_i = 0, \forall i \rightarrow x = 0$$

If $x = 0 \rightarrow x = (0,0,\dots,0,\dots) \rightarrow x_i = 0, \forall i \rightarrow |x_i| = 0 \rightarrow |x_i|^p = 0 \rightarrow \sum |x_i|^p = 0 \rightarrow (\sum |x_i|^p)^{\frac{1}{p}} = 0 \rightarrow \|x\| = 0$

3- Let $x \in \ell^p, \lambda \in R$

$$x = (x_1, x_2, \dots, x_n, \dots)$$

$$\|\lambda x\| = \|\lambda(x_1, x_2, \dots, x_n, \dots)\| = \|\lambda x_1, \lambda x_2, \dots, \lambda x_n, \dots\|$$
$$= \left(\sum |\lambda x_i|^p\right)^{\frac{1}{p}} = \left(\sum |\lambda|^p |x_i|^p\right)^{\frac{1}{p}}$$
$$= \left(|\lambda|^p \sum |x_i|^p\right)^{\frac{1}{p}} = (|\lambda|^p)^{\frac{1}{p}} \left(\sum |x_i|^p\right)^{\frac{1}{p}}$$
$$= |\lambda| \left(\sum |x_i|^p\right)^{\frac{1}{p}} = |\lambda| \|x\|$$

4- *Let* $x, y \in \ell^p, x = (x_1, x_2, \dots, x_n, \dots), y = (y_1, y_2, \dots, y_n, \dots)$

$$\|x + y\| = \|x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, \dots\|$$
$$= \left(\sum |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |y_i|^p\right)^{\frac{1}{p}}$$
$$= \|x\| + \|y\|$$

Example: *Let* $\|.\|: \ell^\infty \to R$ *such that* $\|x\| = sup|x_i|$, *then* $\|.\|$ *is a normed space H.W*

Example:

Let $X = C[0,1] \to R$ *be a function define as* $\|f\| = max\{|f(x)|, 0 \le x \le 1\}, \forall f \in X$

Show that $\|f\|$ *norm on X*

Solution:

1- *Since* $|f(x)| \ge 0, \forall x \in [0,1] \to \|f\| \ge 0$
2- *Let* $\|f\| = 0 \leftrightarrow max\{|f(x)|, 0 \le x \le 1\} \leftrightarrow |f(x)| = 0, \forall x \in [0,1] \leftrightarrow f(x) = 0, \forall x \in [0,1] \leftrightarrow f = 0$
3- *Let* $f \in X, \lambda \in R$

$$\|\lambda f\| = max\{|(\lambda f)(x)|, 0 \le x \le 1\}$$
$$= max\{|\lambda||f(x)|, 0 \le x \le 1\}$$
$$= |\lambda|\, max\{|f(x)|, 0 \le x \le 1\} = |\lambda|\|f\|$$

4- Let $f, g \in X$

$$\|f + g\| = max\{|(f + g)(x)|, 0 \le x \le 1\}$$
$$= max\{|f(x) + g(x)|, 0 \le x \le 1\}$$
$$\le max\{|f(x)| + |g(x)|, 0 \le x \le 1\}$$
$$\le max\{|f(x)|, 0 \le x \le 1\} + max\{|g(x)|, 0 \le x \le 1\}$$
$$= \|f\| + \|g\|$$

Example:

Let $X = C[0,1]$ and $\|.\|: X \to R$ be a function define as $\|f\| = \int_0^1 |f(x)|dx, \forall f \in X$

Show that $\|f\|$ norm on X

H.W

Theorem:

Every normed vector space is a metric space but the converse is not true in general

Proof

Let $(X, \|.\|)$ be a normed space defined $d: X \times X \to R$ by

$$d(x, y) = \|x - y\|, x, y \in X$$

1- Since $d(x, y) = \|x - y\| \ge 0, \forall x, y \in X \ \to d(x, y) \ge 0$

2- *If* $x = y \rightarrow d(x, y) = \|x - x\| = 0$ *if*

$d(x, y) = 0 \rightarrow \|x - y\| = 0 \rightarrow x - y = 0 \rightarrow x = y$

3- $\forall x, y \in X$

$d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$

4- *Let* $x, y, z \in X$

$$x - y = x - z + z - y$$

$$d(x, y) = \|x - y\| = \|x - z + z - y\|$$

$$\leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$$

$\rightarrow d(x, y) \leq d(x, z) + d(z, y)$

Then we get normed space is metric space

The converse is H..W

Prodect Space

Definition:

Let X, Y *be the set, the Cartesian product of* X, Y *denoted by* $X \times Y$ *and defined as follows :*

$$X \times Y = \{(x, y), x \in X, y \in Y\}$$

Then $X \times Y \neq Y \times X$ *and if* $X \neq \emptyset$ *and* $Y \neq \emptyset$ *then* $X \times Y \neq \emptyset$

- *If* X, Y *are two vectors over* F *then we can defined* $\forall (x_1, y_1), (x_2, y_2) \in X \times Y$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ *and* $\lambda(x_1, y_1) = (\lambda x_1, \lambda y_1), \forall \lambda \in F$

Example:

If $(X, \|.\|_1)$ *and* $(Y, \|.\|_2)$ *are two normed spaces then* $(X \times Y, \|.\|)$ *is a normed space such that* $\|(x, y)\| = max\{\|x\|_1, \|y\|_2\}, \forall(x, y) \in X \times Y$

Solution

1- *Since* $\|x\|_1 \geq 0, \forall x \in X$ *and* $\|y\|_2 \geq 0, \forall y \in Y \rightarrow max\{\|x\|_1, \|y\|_2\} \geq 0 \rightarrow \|(x, y)\| \geq 0$

2- $\|(x, y)\| = 0 \leftrightarrow max\{\|x\|_1, \|y\|_2\} = 0 \leftrightarrow \|x\|_1 = 0, \|y\|_2 = 0 \leftrightarrow x = 0, y = 0 \leftrightarrow (x, y) = 0$

3- *Let* $(x, y) \in X \times Y, \lambda \in F$

$\|\lambda(x, y)\| = max\{\|\lambda x\|_1, \|\lambda y\|_2\} = |\lambda|\, max\{\|x\|_1, \|y\|_2\} = |\lambda|\|(x, y)\|$

4- *Let* $(x, y), (z, w) \in X \times Y$

$$(x, y) + (z, w) = (x + z, y + w)$$

$$\|(x, y) + (z, w)\| = max\{\|x + z\|_1, \|y + w\|_2\}$$
$$\leq max\{\|x\|_1 + \|z\|_1, \|y\|_2 + \|w\|_2\}$$
$$\leq max\{\|x\|_1, \|y\|_2\} + max\{\|z\|_1, \|w\|_2\}$$
$$= \|(x, y)\| + \|(z, w)\|$$

Open Ball And Closed Ball

Definition:

Let X *be a normed space and let* $x_0 \in X$, *if* r *is a positive real number then the set* $\{x \in X: \|x - x_0\| < r\}$ *is a called* *open ball and we called x_0 ball center , r ball radius and we denoted the open ball with center x_0 and radius r by $B_r(x_0)$ then we get*

$$B_r(x_0) = \{x \in X;\ \|x - x_0\| < r\}$$

Definition:

Let X be a normed space and let $x_0 \in X$, if r is a positive real number then the set $\{x \in X: \|x - x_0\| \le r\}$ is a called closed ball and we called x_0 ball center , r ball radius and we denoted the closed ball with center x_0 and radius r by $\bar{B}_r(x_0)$ then we get

$$\bar{B}_r(x_0) = \{x \in X;\ \|x - x_0\| \le r\}$$

Remark:

1- *In the special case*
   $B_1(0) = \{x \in X;\ \|x\| < 1\}$ *is called open uint and we called the set*
   $\bar{B}_1(0) = \{x \in X;\ \|x\| \le r\}$ *closed uint*
2- *We can prove that*
   a- $B_r(x_0) = x_0 + rB_1(0)$
   b- $\bar{B}_1(0) = x_0 + r\bar{B}_1(0)$

Example:

If X is a normed space then the open ball and closed ball is a convex set

Solution

Let $B_r(x_0)$ is an open ball with center x_0 and radius r to show that $B_r(x_0)$ is a convex set

Let $x, y \in B_r(x_0), \lambda \in F, 0 \le \lambda \le 1$

Since $x \in B_r(x_0) \to \|x - x_0\| < r$ and

$$y \in B_r(x_0) \to \|y - x_0\| < r$$

Now, $\underline{\lambda x} + (1-\lambda)y - x_0 = \lambda x + (1-\lambda)y \underline{\underline{-x_0}} + \underline{\underline{\lambda x_0}} \underline{-\lambda x_0}$

$$= \lambda(x - x_0) + \underline{(1-\lambda)y - (1-\lambda)x_0}$$
$$= \lambda(x - x_0) + (1-\lambda)(y - x_0)$$

$$\|\lambda x + (1-\lambda)y - x_0\| = \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\|$$
$$\le |\lambda|\|x - x_0\| + |1-\lambda|\|y - x_0\| < \lambda r + (1-\lambda)r$$
$$= \lambda r + r - \lambda r = r$$

$\therefore \|\lambda x + (1-\lambda)y - x_0\| < r$

$$\to \lambda x + (1-\lambda)y \in B_r(x_0)$$

$B_r(x_0)$ is a convex set

$\overline{B}_r(x_0)$ H.W

Definition:

Let A be a subset of a normed space X, we say that A is an open set in X if , $\forall x \in A \exists r > 0$ such that $B_r(x_0) \subseteq A$

Remark:

In a normed space we can prove that

1- *Every open ball is an open set*
2- *Each* $\emptyset, X$ *are open set*
3- *The union of finite or infinite open set is open set*
4- *The intersection of finite open set is open set*
5- *If* $A \subseteq X$ *then* A *is open set iff a equal to union of open set*

Definition:

Let A *be a subset of a normed space* X*, we say that* $x \in A$ *is interior point in* A *if ,* $\exists r > 0$ *such that* $B_r(z) \subseteq A$

The set of all element is interior points denoted by $int(A)$ *or* A°

Remark:

we can prove that

1- $int(A) \subseteq A$
2- $int(A)$ *open set*
3- A *open set iff* $int(A) = A$
4- $int\big(int(A)\big) = int(A)$
5- $int(A) = \cup \{M_i, M_i$ *open set* $M_i \subseteq A\}$ *then* $int(A)$ *largest open set contain* A
6- $int(A) = \{x \in A: \exists r > 0, x + rB_1(0) \subseteq A\}$

Definition:

Let A be a subset of a normed space X, we say that A is an open set in X if complement of A is open set in X

Remark:

In a normed space we can prove that

1- *Every closed ball is an closed set*
2- *Each $\emptyset, X$ are closed set*
3- *The union of finite closed set is closed set*
4- *The intersection of finite or infinite closed set is closed set*

Theorem:

If X normed space, then every set contain only one element is a closed set

Proof

Let $A = \{x\}$ we prove that A closed set

i.e\ to show that A^c is open set

let $y \in A^c \to y \notin A$, let $x \in A$

$\|x - y\| > 0, x \neq y$

Take $\|x - y\| = r\ , r > 0$

Since $\|x - y\| = r \to \|x - y\| \nless r \to x \notin B_r(y)$

$\to A \cap B_r(y) = \emptyset$

$\to B_r(y) \subseteq A^c$

$\rightarrow A^c$ *open set*

$\rightarrow A$ *closed set*

Lemma:

Every finite subset of a normed space is closed set

Proof

Let A *be a sub set ,* A *finite of a normed space in* X

If $A = \emptyset \rightarrow A$ *closed set (by remark)*

If $A \neq \emptyset \rightarrow \exists x_1, x_2, \dots, x_n \in X$ *such that*

$A = \{x_1, x_2, \dots, x_n\}$ *since* x_i *closed set* $\forall i = 1, \dots n \rightarrow A = \bigcup_{i=1}^{n} x_i$ *closed set (by remark)*

Definition:

Let A *be a subset of a normed space, say the point* $x \in X$ *accumulution point or limit point of a set* A *iff* $\forall$ *open set* G *in* X *contain* x *other point* y $y \neq x, y$ *in* A *or if* G *open set in* X *and* $x \in G$ *then* $A \cap \left({}^{G}/_{\{x\}} \right) \neq \emptyset$

The set of all limt point of A *denoted by* A'

Definition:

Let A be a subset of a normed space X, the set $A \cup A'$ is called closure of A and denoted by $\overline{A}$ (i.e\ $\bar{A} = A \cup A'$

Remark:

By definition we can prove

1- $A \subseteq \bar{A}$, $A' \subseteq \bar{A}$
2- $x \in \bar{A}$ *iff* $\forall r > 0\ \exists y \in A \ni \|x - y\| < r$
3- $\bar{A}$ *closed set*
4- A *close set iff* $A = \bar{A}$
5- $\bar{\bar{A}} = \bar{A}$
6- $\bar{A} = \cap \{M_i \setminus M_i \text{closed set}, A \subseteq M_i\}$ $\bar{A}$ *smallest closed set contain* A
7- $\bar{A} = \bigcap_{r>0} (A + r\overline{B_1}(0))$

Theorem:

If M is a subspace of a normed space then $\overline{M}$ is subspace

Convergence in normed space

Convergence in normed space

Definition:

Let X *be a non- empty set, then the function* $f: N \to X$ *such that* $\forall n \in N$ $\exists$ *only one element such that* $f(n) = x_n$ *is called a sequence in* X

We denoted of f *by* $\{x_n\}$ *and id said to be*

$n -$ *terme of sequence*

The range of sequence $\{x_n\}$ *is set* $\{x_n, n \in N\}$

If $x_n = 2(-1)^{n-1}, n \in N$ *is a sequence define on* R *then* $\{x_n\} = \{2(-1)^{n-1}\} = \{2, -2, 2, -2, \dots\}$ *is a sequence*

But $\{x_n, n \in N\} = \{2, -2\}$ *is a range*

Definition:

Let $\{x_n\}$ *be a sequence in a normed space* X *then* $\{x_n\}$ *is said to be converge in* X *if there exist* $x \in X$ *such that for any* $\epsilon > 0, \exists k \in Z^+$ *such that*

$\|x_n - x\| < \epsilon, \forall n > k$

x *is said to be a convergent point*

$$x_n \to x \leftrightarrow \|x_n - x\| \to 0$$

If x_n is non – convergent then said to be divergent

Theorem:

If the sequence $\{x_n\}$ is a convergent in a normed space x then the convergent point is unique

Proof

Let $x_n \to x, and\ x_n \to y$ such that $x \neq y$

And let $\|x - y\| = \epsilon \to \epsilon > 0$

Since $x_n \to x$

$\Rightarrow \exists k_1 \in Z^+$ *such that* $\|x_n - x\| < \frac{\epsilon}{2}, \forall n > k_1$

And since, $x_n \to y$

$\Rightarrow \exists k_2 \in Z^+$ *such that* $\|x_n - y\| < \frac{\epsilon}{2}, \forall n > k_2$

Take $k = max\{k_1, k_2\}$then

$$\|x_n - x\| < \frac{\epsilon}{2}, \|x_n - y\| < \frac{\epsilon}{2}, \forall n > k$$

$$\begin{aligned}\epsilon = \|x - y\| = \|x + x_n - x_n - y\| &= \|(x - x_n) + (x_n - y)\| \\ &= \|-(x_n - x) + (x_n - y)\| \\ &\leq \|x_n - x\| + \|x_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

$\Rightarrow \epsilon < \epsilon$ *(cont #)*

$$\Rightarrow x = y$$

Theorem:

Let $\{x_n\}, \{y_n\}$ *are sequence in a normed space* X *such that* $x_n \to x, y_n \to y$ *then*

1- $x_n + y_n \to x + y$
2- $\lambda x_n \to \lambda x$
3- $\|x_n\| \to \|x\|$
4- $\|x_n - y_n\| \to \|x - y\|$

Proof

1- $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|x_n - y\|$ *since*

$$x_n \to x \Rightarrow \|x_n - x\| \to 0$$

And

$$y_n \to y \Rightarrow \|y_n - y\| \to 0$$

Then

$$\|x_n - x\| + \|x_n - y\| \to 0$$

$$\Rightarrow \|(x_n + y_n) - (x + y)\| \to 0$$

Then $x_n + y_n \to x + y$

2- *H.W*
3- *Since* $|\|x_n\| - \|x\|| \leq \|x_n - x\|$ *and* $\|x_n - x\| \to 0$

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0$$

$$|\|x_n\| - \|x\|| \to 0$$

$$\Rightarrow \|x_n\| \to \|x\|$$

4- $|\|x_n - y_n\| - \|x - y\|| \leq \|(x_n - y_n) - (x - y)\| \leq$
$\|x_n - x\| + \|x_n - y\|$

And

$$x_n \to x \Rightarrow \|x_n - x\| \to 0$$

$$y_n \to y \Rightarrow \|y_n - y\| \to 0$$

$$\Rightarrow \|x_n - x\| + \|x_n - y\| \to 0$$

$$\Rightarrow |\|x_n - y_n\| - \|x - y\|| \to 0$$

$$\Rightarrow \|x_n - y_n\| \to \|x - y\|$$

Definition:

Let $\{x_n\}$*is a sequence in a normed space* X *we say that* $\{x_n\}$ *is a coushy sequence in* X *if* $\forall\epsilon > 0, \exists k \in Z^+$ *such that*

$\|x_n - x_m\| < \epsilon, \forall n, m > k$

Theorem:

Every convergent sequence is a caushy sequence

Proof

Let $\{x_n\}$ *is a converg sequence to* x

$\Rightarrow x_n \to x$

Let $\epsilon > 0$*, since* $x_n \to x$ *and* $\epsilon > 0 \to \frac{\epsilon}{2} > 0$

$\Rightarrow \exists k \in Z^+$ *such that* $\|x_n - x\| < \epsilon, \forall n > k$

$if\ n, m > k$

$$\Rightarrow \|x_n - x_m\| = \|x_n - x_m + x - x\|$$
$$= \|(x_n - x) + (x_m - x)\|$$
$$\leq \|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow \|x_n - x_m\| < \epsilon, \forall n, m > k, k \in Z^+$$

$\Rightarrow \{x_n\}$ *is a coushy sequen*

Definition:

Let $\{x_n\}$ *be a sequence in a normed space* X *we say that* $\{x_n\}$ *is a bounded sequence if there exsists a positive number* M *such that* $\|x_n\| \leq M, \forall n \in Z^+$

Theorem:

If $\{x_n\}$ *is a coushy sequence in a normed space* X *then* $\{x_n\}$ *is a bounded*

Proof

Let $\epsilon = 1 , \epsilon > 0$

Since $\{x_n\}$ *is a coushy sequence*

$\Rightarrow \exists k \in Z^+ \ni \|x_n - x_m\| < 1, \forall n, m > k$

Let $m = n + 1$

$\Rightarrow \|x_n - x_{n+1}\| < 1, \forall n > k$

Since

$|\|x_n\| - \|x_{n+1}\|| < \|x_n - x_m\| < 1$

$\Rightarrow \|x_n\| - \|x_{n+1}\| < 1, \forall n > k$

$\Rightarrow \|x_n\| < 1 + \Rightarrow \|x_{n+1}\|, \forall n > k$

Take $M = max\{\|x_1\|, \|x_2\|, \dots, \|x_n\|, \|x_{n+1}\|, \|x_{n+1}\| + 1\}$

$\Rightarrow \|x_n\| \leq M, \forall n \in Z$

Corollary:

Every converg sequence in a normed space X *is a bounded sequence*

Banach space

Definition:

We say that a normed space X *is a complete space iff for any coushy sequence in* X *is a converge sequence in* X

A complete normed space is said to be banach space

Example:

A space F^n with norm $\|x\| = (\sum_{i=1}^{n}|x|^2)^{\frac{1}{2}}$,

$\forall x = (x_1, x_2, \dots, x_n) \in F^n$ is a banach space

Solution:

1- F^n *normed space (proof???)*
2- *Let $\{x_n\}$ is a cauchy sequence in F^n*

$\Rightarrow x_n \in F^n \Rightarrow x_n = (x_{1n}, x_{2n}, \dots, x_{in})$

Let $\epsilon > 0$, $\exists k \in Z^+$ such that

$\|x_m - x_I\| < \epsilon, \forall m, I > k$

$\|x_m - x_I\|^2 < \epsilon^2$

$\because x_m - x_I = (x_{1m} - x_{1I}, x_{2m} - x_{2I}, \dots, x_{im} - x_{iI})$

$$\Rightarrow \|x_m - x_I\| = \sum |x_{im} - x_{iI}|^2$$

Since

$\|x_m - x_I\|^2 < \epsilon^2, \forall m, I > k$

$\Rightarrow \|x_m - x_I\| < \epsilon, \forall m, I > k$

Then for any $i \in I, \{x_{im}\}$is a cauchy sequence in F

Since F is complete (because F either R or C)

$\Rightarrow$ *for any* $\|x_{im} - x_{iI}\| < \frac{\epsilon}{\sqrt{n}}$

Put $x = (x_1, x_2, \dots, x_n) \Rightarrow x \in F^n$

Let $\epsilon > 0 \exists k \in Z^+$ *for any* $m > k$ *we get*

$$\|x_n - x\|^2 = \sum_{i=1}^{n} |x_{in} - x_i|^2 < \epsilon^2$$

$$\Rightarrow \|x_{in} - x\| < \epsilon$$

$\Rightarrow < x_n >$ *is a converge*

$\Rightarrow F^n$ *complete*

From (1) and (2) we get F^n *is a banach space*

<u>Example: H.W</u>

The space L^p $(1 \leq p \leq \infty)$ *with norm*

$\|x\| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ $, \forall x = (x_1, x_2, \dots) \in L^p$ *is a banach space*

<u>Example:</u>

A space L^∞ *with norm* $\|x\| = sup|x_i|$

$\forall x = (x_1, x_2, \dots) \in L^\infty$ *is a banach space*

Solution

1- L^∞ *is a normed space H.W*
2- *Let* $< x_m >$ *is a cauchy sequence in* L^∞

$\Rightarrow x_m \in L^{\infty}$

$\Rightarrow x_m = (x_{1m}, x_{2m}, \dots, x_{nm}, \dots)$

Let $\epsilon > 0, \exists k \in N \ni$

$$\|x_m - x_L\| < \epsilon, \forall m, L > k$$

$\because x_m - x_L = (x_{1m} - x_{1L}, x_{2m} - x_{2L}, \dots, x_{nm} - x_{nL}, \dots)$

$\|x_m - x_L\| = sup|x_{im} - x_{iL}|$

$\Rightarrow sup|x_{im} - x_{iL}| < \epsilon, \forall m, L > k$

$\Rightarrow |x_{im} - x_{iL}| < \epsilon, \forall m, L > k$ *for any* i

Any $< x_{im} >$ *is a cauchy sequence in* F

Since F *is a complete*

$\Rightarrow < x_{im} >$ *converge in* F

Then $\exists x_i \in F$ *such that*

$< x_{im} >$ *converge to* x_i

Put $x_i = (x_1, x_2, \dots)$

$\|x_{im} - x_i\| < \epsilon, \forall m > k$

$\because x_m \in L^{\infty} \Rightarrow \exists k_m \in R \ni$

$\|x_{im}\| \leq k_m, \forall i$

$x_i = (x_i - x_{im}) + x_{im}$

$$|x_i| \leq |x_i - x_{im}| + |x_{im}| < \epsilon + k_m, \forall m > k$$

$\forall x \in L^{\infty}$

$$\|x_m - x\| = sup|x_{im} - x_i| < \epsilon$$

$\Rightarrow x_m \to x$

$\Rightarrow L^{\infty}$ *is a complete*

Definition:

Let $(X, \|.\|_1)$ *and* $(Y, \|.\|_2)$ *be two normed space , a function* $f: X \to Y$ *is said to be continous at* $x_0 \in X$ *if*

$\forall \epsilon > 0, \exists \delta > 0, \|x - x_0\|_1 < \delta$

$\Rightarrow \|f(x) - f(x_0)\ \|_2 < \epsilon$

Equivelently

$\forall x_n \to x_0$ *in* X

$\Rightarrow f(x_n) \to f(x)$ *in* Y

We say that f *is a continous of each point of* X

Example:

Let X *be a normed space then the function* $f: X \to R$ *such that* $f(x) = \ \|x\|$ *is a continous*

Solution

Let $x_n \to x_0$ *in* X

$\Rightarrow \|x_n - x_0\| \to 0, n \to \infty$

$$|f(x_n) - f(x_0)| = |\|x_n\| - \|x_0\||$$

$$\leq \|x_n - x_0\| \to 0$$

$\Rightarrow |f(x_n) - f(x_0)| \to 0, n \to \infty$

$\Rightarrow f(x_n) \to f(x_0)$

$\Rightarrow f$ *continous at* x_0

$\Rightarrow f$ *continous*

Definition:

Let X, Y, Z *are normed space , we say that* $f: X \times Y \to Z$ *continous at point* $(x_0, y_0) \in X \times Y$ *if* $f(x_n, y_n) \to f(x_0, y_0)$ *,* $\forall x_n \to x_0$ *in* X *and* $\forall y_n \to y_0$ *in* Y

Theorem:

Let X *be a normed space on a felid* F *then*

1- $f: X \times X \to X$ *,* $f(x, y) = x + y, \forall x, y \in X$
2- $f: F \times X \to X$ *,* $g(\lambda y) = \lambda x, \forall \lambda \in F, x \in X$ *are continous function*

Proof

1- *Let* $x_n \to x_0$*,* $y_n \to y_0$

$$\begin{aligned} \|f(x_n, y_n) - f(x_0, y_0)\| &= \|(x_n + y_n) - (x_0 + y_0)\| \\ &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|(x_n - x_0)\| + \|(y_n - y_0)\| \end{aligned}$$

Since $\|(x_n - x_0)\| \to 0, \|(y_n - y_0)\| \to 0$

$\Rightarrow \|(x_n - x_0)\| + \|(y_n - y_0)\| \to 0$

$\Rightarrow \|f(x_n, y_n) - f(x_0, y_0)\| \to 0$

$\Rightarrow f(x_n, y_n) \to f(x_0, y_0)$

$\Rightarrow f$ *continous function in* (x_0, y_0)

2- Let $\lambda_n \to \lambda,$ *in* F $x_n \to x_0$ *in* X

$$\|g(\lambda_n, x_n) - g(\lambda, x_0)\|$$
$$= \|(\lambda_n x_n) - (\lambda x_0)\|$$
$$= \|\lambda_n x_n - \lambda x_0 + \lambda_n x_0 - \lambda_n x_0\|$$
$$= \|\lambda_n(x_n - \lambda_n x_0) + \lambda_n(x_0 - \lambda x_0)\|$$
$$\leq |\lambda_n|\|(x_n - x_0)\| + |x_0|\|(\lambda_n - \lambda)\|$$

Since $|\lambda_n - \lambda| \to 0, |x_n - x_0| \to 0$

$\Rightarrow \|g(\lambda_n, x_n) - g(\lambda, x_0)\| \to 0$

$\Rightarrow g(\lambda_n, x_n) \to g(\lambda, x_0)$

$\Rightarrow g$ *continous function*

Example:

Let X, Y *are normed space ,* $f: X \to Y$ *linear transformation if* f *continous at* 0 *then* f *continous at each point*

Proof

Let $x_n \to x_0$ *in* X

$\Rightarrow x_n - x_0 \to 0$

Since f *continous at* 0

$\Rightarrow f(x_n - x_0) \to f(0)$

Since $f(0) = 0$ *,* f *linear*

$f(x_n - x_0) = f(x_n) - f(x_0)$

$\Rightarrow f(x_n) - f(x_0) \to 0$

$\Rightarrow f(x_n) \to f(x_0)$

$\Rightarrow f$ *continuos at* x_0

$\Rightarrow f$ *continous at each point*

Note:

The set of all linear function from X *in to* Y *will be denoted by* $L(X, Y)$

If $f: X \to X$ *denoted* $L(X)$

Theorem:

Let $(X, \|.\|)$ *and* $(Y, \|.\|)$ *be a normed space and let* $f: X \to Y$ *be a linear function , then* f *is a continous iff* f *is a continous at* 0

Proof

Suppose that f *is a continous at* 0

Let $x_n \to x$ *in* X

$\Rightarrow x_n - x \to 0$

Since f *continous at* 0

$\Rightarrow f(x_n - x) \to f(0)$ *in* Y

Since $f(0) = 0$ *and* f *linear*

$f(x_n - x) = f(x_n) - f(x)$

$\Rightarrow \ f(x_n) - f(x) \to 0$

$\Rightarrow \; f(x_n) \to f(x)$

$\Rightarrow f$ *continuos at* x

<u>The conversely H.W</u>